Developing the

Theory of Everything

Second Edition

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Contents

Nomenclature

$(1, \sigma^1, \sigma^2, \sigma^3, i, i\sigma^1, i\sigma^2, i\sigma^3)$ basis of Cl_3, page 48

$(\mathrm{D}_0,\ \mathrm{D}_1,\ \mathrm{D}_2,\ \mathrm{D}_3)$ mobile basis, page 45

$(a)_n$ $(a)_0 = 1$ $(a)_1 = a$, $(a)_n = a(a+1)\dots(a+n-1)$, see equation (C.58)

(r, θ, φ) spherical coordinates, see equation (C.1)

(V_1, V_2, V_3, V_4) canonical basis in $M_2(\mathbb{C})$, see equation (1.5)

$\alpha = \frac{e^2}{\hbar c}$ fine structure constant, see equation (C.21)

β Yvon-Takabayasi angle, see equation (1.89)

$\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ Christoffel symbols (from contravariance), see equation (4.76)

$\boldsymbol{D}$ Invariant derivative, see equation (4.98)

$\Box = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2$ D'Alembertian, see equation (A.47)

$\Delta = (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2$ Laplacian operator, see equation (A.37)

$\epsilon = \frac{E}{m}$ reduced mass-energy, see equation (C.62)

η left wave, see equation (1.3)

η^1 left wave of the electron, page 54

$\frac{m}{k\mathbf{l}}\mathcal{L}^1 + \frac{m}{k\mathbf{r}}\mathcal{L}^2 + \frac{m}{km_l}\mathcal{L}^3 + \frac{m}{km_r}\mathcal{L}^4$ Lagrangian density, page 118

$\Gamma^{\beta}_{\mu\nu}$ Christoffel symbols (from covariance), see equation (4.18)

$\gamma^j = -\gamma_j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}$ Dirac matrices ($j = 1, 2, 3$), see equation (B.8)

$\gamma_0 = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ Dirac matrices (our choice), see equation (B.8)

$\Gamma_4 = iL_4$, see equation (B.54)

$\Gamma_5 = -iL_5$, see equation (B.54)

$\Gamma_\mu = L_\mu$ $\mu = 0, 1, 2, 3$, see equation (B.54)

κ constant (nonzero integer number), see equation (C.38)

λ magnetic quantum number, see equation (C.44)

$\Lambda_n, \ n = 1, ..., 8$ generator of the $SU(3)_c$ group, see equation (3.45)

$\mathbb{R}$ field of real numbers, page 30

$\mathbb{C}$ field of complex numbers, page 30

$\mathbf{A} := \gamma_\mu A^\mu$ electromagnetic potential (in space-time algebra), see equation (B.15)

$\mathbf{F} = \begin{pmatrix} F & 0 \\ 0 & \widehat{F} \end{pmatrix}$ electromagnetic field (in space-time algebra), see equation (B.16)

$\mathbf{i} = \gamma_{0123} = \gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma_5$, see equation (B.52)

$\mathbf{l}$ left mass term, see equation (1.147)

$\mathbf{m} = \begin{pmatrix} \mathbf{l} & 0 \\ 0 & \mathbf{r} \end{pmatrix}$ matrix mass term, see equation (1.147)

$\mathbf{n}$ principal quantum number, page 64

$\mathbf{n} = |\kappa| + n$ (n: degree of radial polynomial functions), page 65

$\mathbf{r}$ right mass term, see equation (1.147)

$\mathcal{F}(\mathbb{R}^4, \mathbb{C})$ set of all ψ (wave functions), page 24

$\mathcal{F}(\mathbb{R}^4, \mathbb{C}^4)$ set of all ψ (Dirac wave functions), page 25

$\mathcal{G}$ group of 2×2 complex matrices $\underline{M}$ such as $|\det(\underline{M})| = 1$, see equation (1.39)

$\mathcal{L}$ Lagrangian density, see equation (1.133)

$\mathcal{L}_+^\uparrow$ restricted Lorentz group, page 34

$\mathcal{L}_q^+;\ \mathcal{L}_q^-$ Lagrangian densities (quarks), see equation (3.133)

$\mathrm{b};\ \mathrm{w}^j;\ \mathrm{h}_j^k$ potential space-time vectors, see equation (3.87)

$\mathrm{D}_0 = \mathrm{J}$ probability current, see equation (1.93)

$\mathrm{D}_1 = \phi\sigma_1\phi^\dagger$ first new current, see equation (1.93)

$\mathrm{D}_2 = \phi\sigma_2\phi^\dagger$ second new current, see equation (1.93)

$\mathrm{D}_3 = \mathrm{K}$ second current, see equation (1.93)

$\mathrm{D}_L^1 = L^1\widetilde{L}^1$ left current, see equation (1.103)

$\mathrm{D}_L^8 := \widetilde{L}^8 L^8$ current of the left neutrino-monopole, page 91

$\mathrm{D}_R^1 = R^1\widetilde{R}^1$ right current, page 44

$\mathrm{D}_R^8 := \widetilde{R}^8 R^8$ current of the right neutrino-monopole, page 91 j electric current (space-time vector), see equation (A.58)

$\mathrm{J} = \mathrm{J}^\mu \sigma_\mu$ probability current, page 41

$\mathrm{J}^\mu = \overline{\psi}\gamma^\mu\psi$ densities, components of J, page 41

k magnetic current (space-time vector), page 223

$\mathrm{K}^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi$ components of the K current, see equation (1.87)

K_l left minus right current, see equation (4.177)

$\mathrm{v} = \frac{1}{\rho}\mathrm{J}$ reduced velocity, see equation (1.164)

$\mathrm{x} = \mathrm{x}^\mu\sigma_\mu$ general element in space-time, see equation (A.43)

$\nabla = \partial_0 - \vec{\partial}$ first differential operator in space-time, see equation (A.46)

$\nu = E/h$ frequency, see equation (1.205)

$\Omega = r^{-1}(\sin\theta)^{-\frac{1}{2}}S$ dilator, see equation (C.3)

$\Omega_1 = \overline{\psi}\psi$ relativistic invariant, page 41

$\Omega_2 = -i\overline{\psi}\gamma_5\psi$ second relativistic invariant, see equation (1.88)

$\overline{\psi} = \psi^\dagger\gamma_0$ Dirac conjugate, page 41

$\overline{A} = \widehat{A}^\dagger$ A bar, see equation (A.32)

$\partial_\mu = \frac{\partial}{\partial x^\mu}$ partial derivative, see equation (1.10)

$\phi = \sqrt{2}\begin{pmatrix}\xi_1^1 & -\eta_2^{1*}\\ \xi_2^1 & \eta_1^{1*}\end{pmatrix}$ wave of the electron, page 38

$\phi_p = -\phi_e\sigma_1$ wave of the positron (in Cl_3), page 61

$\boldsymbol{\partial} = \gamma^\mu\partial_\mu$ Dirac differential operator, see equation (B.9)

$\boldsymbol{\partial}_\nu = \frac{\partial}{\partial X^\nu} = \mathrm{D}^\mu_\nu\partial_\mu$ Dirac operator, see equation (4.18)

$\Psi : \phi \mapsto \phi_e$ wave with value: operator on Cl_3, see equation (2.1)

$\psi = \psi(x,y,z,t)$ wave function (function of space and time with complex value), see equation (1.1)

Ψ_b wave d(blue)+ u(blue), see equation (2.6)

Ψ_g wave d(green)+ u(green), see equation (2.6)

Ψ_l wave electron + neutrino-monopole, see equation (2.6)

$\Psi_L = \Psi_L^1 + \Psi_L^8$ left part of the lepton wave, page 104

ψ_p Dirac wave of the positron, see equation (1.140)

$\Psi_q = \begin{pmatrix} i\Psi_b & \Psi_r + \Psi_g\\ \Psi_r - \Psi_g & -i\Psi_b\end{pmatrix}$ quark wave, see equation (3.3)

Ψ_r wave d(rot)+ u(rot), see equation (2.6)

ρ main relativistic invariant, see equation (1.91)

ρ_l generalization of ρ in the lepton case, see equation (2.26)

σ_μ Pauli matrices, see equation (1.4)

$\sigma_{21} = \sigma_2\sigma_1$ is a 2-vector in Cl_3, page 39

θ_W Weinberg–Salam angle, see equation (2.201)

$\underline{D}$ gauge-invariant derivative, see equation (3.50)

$\underline{\mathrm{J}} := \frac{m}{\mathbf{l}}\mathrm{D}^1_L + \frac{m}{\mathbf{r}}\mathrm{D}^1_R$ barycentric current, see equation (1.302)

$\vec{\partial}'$ $(= \sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi)$, see equation (C.3)

$\vec{\partial} = \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix}$ main differential operator in Cl_3, see equation (A.35)

$\vec{\partial}\cdot\vec{u}$ divergence of $\vec{u}$, see equation (A.39)

$\vec{\partial}\times\vec{u}$ rotational of $\vec{u}$, see equation (A.39)

$\vec{u}\cdot\vec{v}$ scalar product, see equation (A.2)

$\vec{u}\times\vec{v}$ vector product (or cross product), page 217

$\vec{\mathrm{grad}}\, a = \vec{\partial}a$ gradient of the scalar a, see equation (A.39)

$\vec{E}$ electric field, page 221

$\vec{H}$ magnetic field, page 221

$\widehat{\nabla} = \partial_0 + \vec{\partial}$ second differential operator in space-time, see equation (A.46)

$\widehat{A} = A_1 - A_2$ A hat, see equation (A.30)

$\widetilde{L}^{3+n} = \widetilde{\phi}^{3+n}\frac{1-\sigma_3}{2}$ $n = 2, 3, 4$, see equation (3.8)

$\widetilde{R}^{3+n} = \widetilde{\phi}^{3+n}\frac{1+\sigma_3}{2}$ $n = 2, 3, 4$, see equation (3.8)

ξ right wave, see equation (1.3)

ξ^1 right wave of the electron, page 54

$A \mapsto \widetilde{A}$ reversion, see equation (A.9)

$A,\ B,\ C,\ D$ functions of r (radial variable), see equation (C.37)

$A^\dagger = (A^*)^t$ adjoint(conjugate transposed), see equation (A.29)

$A_1 = a + i\vec{v}$ even part of $A = a + \vec{u} + i\vec{v} + ib$, page 219

$A_2 = \vec{u} + ib$ odd part of $A = a + \vec{u} + i\vec{v} + ib$, page 219

$a_n, n = 1, 2, ..., 6$ invariant densities, see equation (2.23)

B chiral potential (space-time vector), page 223

$C_{\mu\nu}$ curvature field, see equation (4.176)

Cl_2 Clifford algebra of the Euclidean plane, page 215

Cl_3 Clifford algebra of 3-dimensional space, page 216

Cl_3^* group of invertible elements in Cl_3, page 30

Cl_3^+ even sub-algebra of Cl_3 (quaternion field), page 217

$Cl_{1,3}$ space-time algebra, see equation (B.3)

$Cl_{1,3}^+$ even space-time subalgebra, see equation (B.6)

$d := \frac{\mathbf{l}-\mathbf{r}}{2}$ mass difference, page 51

D^* group of similitudes R, page 32

$D_{\mathrm{x}} : X \mapsto \mathrm{x} = \phi X \phi^\dagger$ induced similitude, see equation (1.279)

$d_\mu^L := -i\partial_\mu + qA_\mu + \mathbf{l}\mathrm{v}_\mu$ covariant derivative (left wave), see equation (1.198)

$d_\mu^R := -i\partial_\mu + qA_\mu + \mathbf{r}\mathrm{v}_\mu$ covariant derivative (right wave), see equation (1.198)

$f : M \mapsto R$ homomorphism: dilator $\mapsto$ similitude, page 157

$F = \vec{E} + i\vec{H}$ electromagnetic field, page 221

$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ electromagnetic field, see equation (1.300)

$G_{\mu\nu} := \partial_\mu \mathrm{v}_\nu - \partial_\nu \mathrm{v}_\mu$ gravitational field, see equation (1.300)

$GL(2,\mathbb{C}) = Cl_3^*$ group of endomorphisms on $\mathbb{C}^2$, page 35

H Hamiltonian, see equation (1.1)

$i = \sigma_1\sigma_2\sigma_3$ is a 3-vector in Cl_3, page 39

$i_1 = \sigma_{23}$ $(i_1^2 = -1)$, see equation (C.3)

$i_2 = \sigma_{31}$ $(i_2^2 = -1)$, see equation (C.3)

$i_3 = \sigma_{12}$ $(i_3^2 = -1)$, see equation (C.3)

j kinetic momentum number $(J^2\phi = j(j+1)\phi))$, see equation (C.45)

$J^2 = J_1^2 + J_2^2 + J_3^2$ kinetic momentum operator, see equation (C.43)

J_3 kinetic momentum operator, see equation (C.42)

$J_l = \mathrm{D}_R^1 + \mathrm{D}_L^1 + \mathrm{D}_R^8 + \mathrm{D}_L^8$ lepton current, see equation (2.20)

$k = 1.140815(25) \times 10^{-80}\mathrm{s}^2\mathrm{kg}^{-1}$ (constant), see equation (1.344)

$L^1 = \sqrt{2}\begin{pmatrix} \eta_1^1 & 0 \\ \eta_2^1 & 0 \end{pmatrix}$ left part of the ϕ wave, page 38

L^8 left wave of the neutrino-monopole, page 90

$L^n = \phi^n \frac{1-\sigma_3}{2}$ $n = 2, 3, 4$, see equation (3.8)

$L_4 = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}$, see equation (B.51)

$L_5 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$, see equation (B.51)

$L_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}$ $(\mu = 0, 1, 2, 3)$, see equation (B.51)

$l_a = 1.38068(3) \times 10^{-36}\mathrm{m}$ absolute length, see equation (1.344)

$m := \frac{m_0 c}{\hbar}$ m_0 is the proper mass, see equation (1.2)

M dilator (general element in Cl_3), page 31

M_ϕ $SL(2, \mathbb{C})$ part of the electron wave, see equation (1.157)

$m_a := \frac{\mathbf{l}+\mathbf{r}}{2}$ arithmetic mean , see equation (1.155)

$m_g = \sqrt{\mathbf{lr}}$ geometric mean, see equation (1.180)

$M_n(\mathbb{C})$ set of $n \times n$ complex matrices

$m_{abs} = 1.85921(4) \times 10^{-9}\mathrm{kg}$ absolute mass, see equation (1.344)

$N = s + v + B + p_v + p_s$ general element in space-time algebra, see equation (B.3)

$P : A \mapsto \widehat{A}$ parity transformation, see equation (A.30)

$P : M \mapsto \widehat{M}$ main automorphism in Cl_3 (parity), page 39

P_+, P_- projectors, see equation (2.43)

$P_\mu,\ \mu = 0, 1, 2, 3$ projectors, see equation (2.47)

r ratio of the similitude R, page 31

$R : \mathrm{x} \mapsto \mathrm{x}' = M\mathrm{x}M^\dagger$ similitude, page 31

$R^1 = \sqrt{2}\begin{pmatrix} \xi_1^1 & 0 \\ \xi_2^1 & 0 \end{pmatrix}$ right part of the ϕ wave, page 38

R^8 right wave of the neutrino-monopole, page 90

$R^n = \phi^n \frac{1+\sigma_3}{2}$ $n = 2, 3, 4$, see equation (3.8)

R^μ_ν real 4×4 matrix of the similitude R, page 32

$S = e^{-\frac{\varphi}{2} i_3} e^{-\frac{\theta}{2} i_2}$ rotator, see equation (C.3)

$S^{\mu\nu} = i\overline{\psi}\gamma^\mu\gamma^\nu\psi$ electric-magnetic momentum densities, see equation (1.86)

$S_0 = \phi\sigma_0\overline{\phi}$ also equals to a_1 and to $\rho e^{i\beta}$, see equation (1.101)

$S_3 = \phi\sigma_3\overline{\phi}$ space-time 2-vector (6 densities), see equation (1.99)

$S_\mu = \phi\sigma_\mu\overline{\phi}$ (20 densities: 8 old, 12 new), page 43

$SL(2,\mathbb{C})$ group of 2×2 M matrices such as $\det(M) = 1$, page 30

$SO(3)$ group of rotations in 3-dimensional space, page 30

$SU(2)$ subgroup of unitary elements in $SL(2,\mathbb{C})$, page 30

T; $T^{n\mu}_{L\lambda}$ $T^{n\mu}_{R\lambda}$ $T^{3+n\mu}_{L\lambda}$ $T^{3+n\mu}_{R\lambda}$ energy–momentum tensors (quarks), see equation (3.150)

T^{μ}_{ν} energy–momentum tensor density (Tetrode's tensor), see equation (1.200)

$t_a = 4.60545(10) \times 10^{-45}$s absolute duration, see equation (1.344)

$U(1) \times SU(2)$ electroweak gauge group, page 89

U, V functions of θ, see equation (C.37)

V^{μ}_{λ} non-interpreted tensor of O. Costa de Beauregard, page 118

X_{μ} non Lagrangian term, see equation (4.135)

Y_{μ} Lagrangian term, see equation (4.135)

$Z'^0 := \frac{Z^0}{\sqrt{3}}$ boson Z'^0, see equation (2.214)

Introduction

During the last century, gravitation has been understood as space-time curvature thanks to general relativity (GR), mainly developed by Einstein. Quantum physics has been built as a theory of gauge invariant fields, known as the Standard Model (SM) of quantum physics. However, the aim of all theoretical physics, a true unification of these separated parts of physics, the "Theory of Everything," has not yet been achieved. This ToE is also the end goal of our work.

Many attempts have been made in recent decades, usually beginning with quantum physics and aiming to include gravitation. "Developing the Theory of Everything" also starts from quantum mechanics, but in an entirely new way using a fully relativistic formulation of quantum mechanics, as explained in Chapter 1. This new approach allows us to understand the true reason for the quantization of action, as well as the true nature of light and the electromagnetic field, which turns out to be simply the momentum–energy of the quantum wave. Novelties also arise here from the same minimal mathematical tools used for both SM and GR: the Cl_3 algebra described in Appendix A and the $\mathrm{End}(Cl_3)$ algebra in Appendix B. The necessity for this tool in GR comes from Whitney's theorem: space-time being a 4-dimensional manifold, an 8-dimensional linear space is sufficient to obtain an embedding, and moreover Cl_3 is $\mathbb{R}^8$ as linear space and as topological space.

The use of these algebras in physics began as early as 1927 with the Pauli algebra, which is Cl_3, and then with the Dirac algebra in 1928 in the frame of what is now known as "first quantization." This step has been followed by a second one, "field quantization" (electromagnetic field, boson fields of electroweak and strong interactions). Half a century ago, D. Hestenes rebuilt first quantization with the mathematical tool of space-time algebra [73, 74, 75, 76, 78, 79]. We first began our research [12] in this framework, but since then have introduced further developments: First, the main novelty has been the use of an improved wave equation which Chapter 1 explains. Second, the natural geometric framework of Dirac theory is not space-time algebra, but rather the Cl_3 algebra. This was not thought possible before our work [15, 16, 18, 19, 20, 21]. This algebra, first promoted by W. Baylis [3], is isomorphic to the even part of Hestenes' $Cl_{1,3}$ algebra.

This restricted framework allowed us to see the Dirac theory as incomplete: many more tensor densities are available, respecting the relativistic link between spinors and tensors. Third, it was necessary to understand the Cl_3^* Lie group as the invariance group of all laws of electromagnetism, the quantum wave of the electron included [18, 20, 21, 22, 23].

These three steps have allowed us to build a completely relativistic theory of electroweak interactions, generalizing our improved Dirac equation. This wave equation uses proper mass terms: we are able to do precisely what was impossible in the first theory of weak interactions [107]. The improved equation is generalized to a wave equation for all fermions and antifermions of the first generation [25, 26, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 39, 45, 46, 47, 50]. Explaining our previous steps in [47] we obtained new results, like the Lorentz force acting on the whole lepton wave (magnetic monopole included). We also obtained an equation for the complete wave of colored quarks. Our wave equations give a reduced velocity vector for the probability current. This allows us to partially mask the crossing between right and left spinors in proper mass terms. Consequently we were able to express the wave equations in a recursive manner that gives the gauge field's properties. And above all we obtained **the quantization of kinetic momentum** [1] with the expected $\hbar/2$ value, both for the electron, the proton and the neutron with their three colored quarks. We also explained the double logical link between the Lagrangian density and the wave equation.

About quarks which make up protons and neutrons of the atomic nucleus we must recall this: the charge of quarks ($+2/3$ of the positron charge for the u quark and $-1/3$ of the positron charge for the d quark) is measured with extraordinary precision. This is undoubtedly the best experimental result in all physics: if there should be but a tiny departure from these values, a non-ionized atom could not be neutral and electrostatic forces between atoms should be much stronger than gravitational forces. Matter could not then build stars and planets. It is thus very important to satisfy these $+2/3$ and $-1/3$ values and this will be thoroughly explained in the book. The Standard Model such as described here does not need to assign charge values: these values are the mere consequence of constraints imposed by relativistic and gauge invariance. About quarks, the Standard Model tells us several properties that must be explained. First, there is the existence of three and only three color states for each quark – we explain these three. Second, fermions also exist in three "generations" that are very similar: Here as well our explanation is simple, as these three also come from the dimension of space. Thus we do not expect a fourth generation, except for a fourth neutrino, and we explain why. This fourth neutrino may be stable and identical to its antiparticle.

Numerous attempts at unification were made by extending the restricted

1. The use of the word "kinetic" when English language expects "angular" does not come from an error of translation; it is the correction of an unfortunate linguistic deficiency (explained in 1.1.2).

framework of the Minkowski space-time and expressing the Dirac equation in curved space-time. This is today considered a completely solved problem, both via the older formalism with the Dirac matrices and via the space-time algebra used by Hestenes and his Cliffordian school [5][6][9][10]. But all these attempts fundamentally confuse two Lie groups: We explain this in Chapter 1. ToE needing absolute mathematical rigor, it is impossible to build the theory on the quicksand of these previous undertakings. For instance, we find that we certainly need a separate space-time manifold; we also absolutely need the $SU(2)$ group in nonrelativistic quantum mechanics and the $SL(2,\mathbb{C})$ group in relativistic quantum mechanics.

What we propose here follows many previous attempts. Weyl's unified theory was based on the notion of gauge that he introduced[106]. This attempt at a unified theory was equivalent to the use of a similitude group as a local invariance group. Weyl's theory was developed at a time when neither the gauge invariance of the quantum wave nor the chirality of weak interactions could be accounted for. Moreover, the one-parameter Lie group generated by the ratio of similitude is the multiplicative group of positive real numbers when a $U(1)$ group is needed for the electric gauge invariance.

Another major attempt was Penrose's theory of twistors [97][98]. Since this theory also began from the Dirac equation and from left and right spinors of the Dirac theory, there are numerous connections between twistor theory and what we study here. Nevertheless, we start in a very different direction: Penrose's aim was the generalization of the quantum wave within a mathematical framework vast enough to contain both nonrelativistic quantum mechanics, Hermitian linear spaces, generalized probabilities, and the gravitation of GR. In short his aim was the quantization of gravitation. For our part, we do not start from the Hamiltonian form of nonrelativistic quantum mechanics: we begin with only the fermionic part of the Lagrangian density in the Standard Model. This is also due to the necessity for a total logical coherence between the Standard Model and general relativity.

The most important attempts at unification were worked out by Einstein from 1917 until his death [106]. He tried various possible paths to unify electromagnetism, gravitation and quantum physics. It is one of his paths that we develop here in Chapter 4, a space-time manifold with torsion. We also explain why he could not fully develop this approach himself, when quark properties and the importance of chiral waves were yet unknown. Moreover, Einstein could not foresee the inclusion of the space-time manifold as the self-adjoint part of the Cl_3^* Lie group.

Many other attempts, which were very popular thirty years ago, were based on the use of numerous supplementary dimensions of space-time, like string and superstring theories. The starting point of these theories, aiming for a theory of everything, is a greater gauge group: a simple one unifying electromagnetism, weak interactions and strong interactions, like $SU(5)$ or $SO(10)$. We do not follow this still-popular path. Such attempts at grand unified theories have only led to false predictions, like the non-conservation of the baryonic quantum number or the existence of many new particles never observed. This grand unification could not be our starting point: the gauge group of the Standard Model is actually embedded in a smaller gauge group, the $SO(8)$ group which is the natural invariance group of Cl_3, which is 8-dimensional. And this restriction is useful: it gives the reason for the difference between quarks and leptons which do not see strong interactions.

So our research path is new: supplementary degrees of freedom that we use do not come from a greater space-time. The space-time of special relativity (SR) and the tangent space-time used in general relativity are the self-adjoint 4-dimensional part of Cl_3, which is an 8-dimensional linear space on $\mathbb{R}$. Its multiplicative group Cl_3^* which generalizes the $SU(2)$ group of quantum mechanics is also 8-dimensional: here we thus obtain two supplementary dimensions in comparison with the Lorentz group, which is 6-dimensional. Moreover, the $\mathrm{End}(Cl_3)$ group is a 64-dimensional group. It is also a ring, containing Cl_3^* as subring, extensive enough to describe quantum waves of all objects, particles and antiparticles, of each generation. Moreover, multiplication in $\mathrm{End}(Cl_3)$ is a generalization of multiplication in $Cl_3^* = \mathrm{End}(\mathbb{C}^2)$.

This research path does not need new particles; we hence respect and corroborate the Standard Model. The only possible objects that may be added to the ones known are right-handed neutrino waves. We study these complete neutrinos which may also be called magnetic monopoles.

The main reason why we are able to add gravitation to the three other kinds of interactions, is that we use a nonlinear term of proper mass in all wave equations. The Weinberg-Salam model could not obtain such mass terms.

Chapter 1 is devoted to the electron in the Cl_3 algebra framework. Most of the novelties that we add to the Standard Model are presented there, such as the true number of densities, improved wave equation, extended relativistic invariance, double energy-momentum tensor, and the link between these tensors and electromagnetic field. That chapter is the only one which does not seem to be concerned by "second quantization". There we use the notation of first quantization, with experimental results on energy levels obtained in this framework. The main change from the first edition is that we use a mass term for each left and right part of the fermionic wave.

Chapter 2 explains how passing from the Cl_3 algebra to the $\mathrm{End}(Cl_3)$ algebra is equivalent to second quantization for the fermionic part of the Standard Model. Through this algebra we satisfy the decomposition of the full wave function into sixteen parts, eight left ones and eight right ones. This second chapter also studies weak interactions mixing the electron with the electron neutrino. The neutrino wave is incorporated in the wave function of a leptonic magnetic monopole: this is the only possibility of extending the fermionic wave function as allowed by the Standard Model. We also fully explain the origin of the extremal principle and of the quantization of action. The study is extended in Chapter 3 to weak and strong interactions of quarks. We generalize the form-invariant derivative. This derivative simplifies the part of weak interactions for the quark wave function.

Chapter 4 incorporates gravitation into quantum physics such as described in the preceding chapters. The formulation of general relativity as the equality between two tensors is extended to be an equality between two connections of the space-time manifold. We explain whence come both the Pauli exclusion principle and the equivalence principle at the basis of general relativity. The global structure of space-time both accounts for the EPR paradox and gives a cosmic expansion with the most recent estimate of the beginning of the acceleration.

Afterwards we present our conclusions. That chapter includes many items that we cannot elaborate further in this introduction. There we also explain why we now change the title from “a” to “the” theory of everything.

The most technical parts are placed in four appendices. The presentation of the tools of Clifford algebra comprises Appendices A and B. There we show in a detailed and basic manner the algebras used in previous chapters. The resolution of the Dirac equation for the hydrogen atom is thoroughly worked out in Appendix C. Various calculations form Appendix D.

Chapter 1

The electron wave with spin 1/2

First we present the usual matrix framework, the quantum wave of the electron and its wave equation. We study the wave equation in the Clifford algebra of space and in the Clifford algebra of space-time. We study tensor densities of the linear wave equation (the Dirac equation). The form invariance of the wave equation is extended to the multiplicative group of the Clifford algebra of space. The relativistic invariance introduces left and right parts of the wave. We simplify the Lagrangian density from which the wave equation comes, and we study an improved wave equation. A dual logical link exists between the wave equation and Lagrangian density. The electric gauge invariance is not changed. A second gauge invariance and a second conservative current appear. Gauge invariance and form invariance are compatible with mass terms. We coherently set out the normalization of the wave, the charge conjugation, the solutions for the hydrogen atom and the Pauli principle. We study the recursion of the improved wave equation and its consequences. We introduce the notions of numeric dimension and double space-time. We study the energy–momentum vector and the dynamics of two energy–momentum tensors. This gives the Lorentz force for the electron. We identify a direct link between the electromagnetic field and momentum–energy tensors of the electron wave.

1.1 The wave equation of the electron

In 1926 two major breakthroughs were made about the electron: the discovery of the electron spin, which means that the electron is a little magnet, even at rest, and the formulation of a wave equation by Erwin Schrödinger. This equation reads in the notation of 1934 [54] as:

$$\frac{h}{2\pi i}\frac{\partial\psi}{\partial t} = H(\psi), \tag{1.1}$$

where $\psi = \psi(x, y, z, t)$ is a complex number, for each value of x, y, z, t, and h is the Planck constant. The wave of the Schrödinger equation is then a function with partial derivatives from $\mathbb{R}^4$ into $\mathbb{C}$. This wave equation is linear. Functions that are solutions of the wave equation form a linear subspace of the linear space $\mathcal{F}(\mathbb{R}^4, \mathbb{C})$. Most of the concepts of quantum physics so far have come from the study of the Hamiltonian H included in the wave equation. Since we will not use this wave equation, we now proceed with the rest of the story of quantum mechanics.

The electron is also a magnet; this is the origin of the properties of permanent magnets that we use daily. So we must account for this magnet. After Pauli's first attempt to explain these magnetic properties, Dirac made use of and carried forward Pauli's attempt by coming up with another wave equation, only a few months later: this wave equation was published as early as 1928 [61, 62]. Ninety years later, we can present this wave equation (in semi-modern notation, and with the usual Einstein summation convention) as follows [1]:

$$0 = [\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi;\ q := \frac{e}{\hbar c};\ \hbar := \frac{\mathrm{h}}{2\pi};\ m := \frac{m_0 c}{\hbar}. \tag{1.2}$$

The four A_μ are the components of the space-time vector called the exterior electromagnetic potential [2] that is created by other charges; e is the charge of the electron and m_0 is the proper mass [3]. We must see the great difference between the ψ of the Schrödinger equation and the ψ of the Dirac equation expressed as:

$$\psi := \begin{pmatrix}\xi\\ \eta\end{pmatrix};\ \xi := \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix};\ \eta := \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}, \tag{1.3}$$

because now the $\xi_j = \xi_j(x, t)$ and the $\eta_j = \eta_j(x, t)$ play the same role as functions of space and time coordinates with value in the complex field. The Dirac wave is hence a function with derivatives from $\mathbb{R}^4$ into $\mathbb{C}^4$. [4] This

1. Most modern presentations use a system of "natural" units where $c = 1$ and $\hbar = 1$. We will see in 1.5.4 why we cannot use the $\hbar = 1$ convention.

2. We will see in Chapter 3 how this potential is not exterior, but dependent on the wave.

3. The wave equation of the electron always includes a mass term and a charge term. This equation is too often presented without its charge term, as if the electric interaction could be removed and restored at will. No physical process allows us to change or omit the charge since this electric charge is quantized.

4. We may also consider the Dirac wave as a set of four functions from $\mathbb{R}^4$ into $\mathbb{C}$. Thus, even if the usual term for the wave of quantum mechanics is "wave function", here we use four functions and thus the general term "wave" seems more appropriate in the Dirac theory.

wave equation is also linear. The solutions are thus elements of a linear subspace of the set of functions $\mathcal{F}(\mathbb{R}^4, \mathbb{C}^4)$. The Dirac equation needs to choose four suitable γ^μ matrices. Our choice is [5] :

$$\begin{aligned}
&\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ \gamma_j := \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \\
&\sigma^j = -\widehat{\sigma}^j = \widehat{\sigma}_j := -\sigma_j,\ j = 1, 2, 3, \qquad (1.4) \\
&\gamma_0 = \gamma^0 := \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}\ ;\ I_2 = \sigma_0 = \sigma^0 = \widehat{\sigma}^0 = \widehat{\sigma}_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

The σ_μ matrices and their products generate an 8-dimensional algebra on $\mathbb{R}$ which is 4-dimensional on $\mathbb{C}$, named the Pauli algebra or the 2×2 matrix algebra: $M_2(\mathbb{C})$. The choice of the Pauli matrices is fixed by the intrinsic basis (V_1, V_2, V_3, V_4) of $M_2(\mathbb{C})$ where the projectors V_n satisfy:

$$\begin{aligned}
V_1 &:= \frac{1}{2}(\sigma_0 + \sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};\ V_2 := \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
V_3 &:= \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};\ V_4 := \frac{1}{2}(\sigma_0 - \sigma_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (1.5)
\end{aligned}$$

With these usual matrices, ξ is the **right** part and η is the **left** part of the wave, because in the Dirac theory it is the γ_5 matrix that allows us the definition of projectors on the right and left parts of the wave: [6]

$$\gamma_5 := i\gamma_1\gamma_2\gamma_3\gamma_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix};\ \frac{1+\gamma_5}{2}\psi = \begin{pmatrix} \xi \\ 0 \end{pmatrix};\ \frac{1-\gamma_5}{2}\psi = \begin{pmatrix} 0 \\ \eta \end{pmatrix}. \qquad (1.6)$$

The Dirac matrices are not uniquely defined. The Dirac theory easily proves that any other choice satisfies

$$\gamma'^\mu = M\gamma^\mu M^{-1};\ \psi' = M\psi, \qquad (1.7)$$

where M is a 4×4 fixed invertible matrix. This always allows us to come back to our choice (1.4). This choice is convenient both for the resolution of the wave equation in the case of the hydrogen atom [14] [36] and for an electron at high velocity. On the contrary, for the study of an electron with low velocity and for deriving the Pauli equation, the initial choice of γ'_μ matrices was [12]:

$$M = M^{-1} := \frac{1}{\sqrt{2}}(\gamma_0 + \gamma_5);\ \gamma'_0 = M\gamma_0 M = \gamma_5;\ \gamma'_j = M\gamma_j M = -\gamma_j, \qquad (1.8)$$

5. The meaning of the notation $\widehat{\sigma}$ is explained in A.3.3.

6. With the Pauli wave equation the usual terms are "left-handed" and "right-handed," because the Pauli wave is a mixing between the left – right and the up – down projectors. "Left" and "right" are more appropriate in the Dirac theory, because these terms match the column names of 2×2 Pauli matrices.

for $j = 1, 2, 3$. We then have:

$$\psi' = M\psi = \begin{pmatrix} \psi'_1 \\ \psi'_2 \\ \psi'_3 \\ \psi'_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 + \eta_1 \\ \xi_2 + \eta_2 \\ \xi_1 - \eta_1 \\ \xi_2 - \eta_2 \end{pmatrix}; \ \chi := \begin{pmatrix} \psi'_1 \\ \psi'_2 \end{pmatrix}; \ \omega := \begin{pmatrix} \psi'_3 \\ \psi'_4 \end{pmatrix}. \tag{1.9}$$

Using matrices (1.4) the Dirac equation (1.2) reads:

$$0 = \begin{pmatrix} im & \sigma^\mu(\partial_\mu + iqA_\mu) \\ \widehat{\sigma}^\mu(\partial_\mu + iqA_\mu) & im \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}; \ \partial_\mu := \frac{\partial}{\partial x^\mu}; \ x^0 := ct. \tag{1.10}$$

This is equivalent to the system:

$$\begin{aligned} 0 &= \sigma^\mu(\partial_\mu + iqA_\mu)\eta + im\xi, \\ 0 &= \widehat{\sigma}^\mu(\partial_\mu + iqA_\mu)\xi + im\eta. \end{aligned} \tag{1.11}$$

The mass term of the wave equation for the ξ part of the wave contains η, while the mass term of the wave equation for the η part of the wave contains ξ. This crossing of terms forbids the use of either ξ or η alone. Next, to get the true Pauli equation it is necessary to break the space-time symmetry of these equations by:

$$\begin{aligned} 0 &= \partial_0\eta + \vec{\partial}\eta + iq(A_0 - \vec{A})\eta + im\xi; \ \vec{\partial} = \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3, \\ 0 &= \partial_0\xi - \vec{\partial}\xi + iq(A_0 + \vec{A})\xi + im\eta; \ \vec{A} = A^1\sigma_1 + A^2\sigma_2 + A^3\sigma_3. \end{aligned} \tag{1.12}$$

Multiplying by i we get the Hamiltonian form

$$\begin{aligned} -i\partial_0\eta &= i\vec{\partial}\eta - q(\mathrm{A}_0 - \vec{\mathrm{A}})\eta - m\xi, \\ -i\partial_0\xi &= -i\vec{\partial}\xi - q(\mathrm{A}_0 + \vec{\mathrm{A}})\xi - m\eta. \end{aligned} \tag{1.13}$$

But this Hamiltonian form does not have the true properties of the Hamiltonian operator of the Schrödinger or the Pauli wave equations [7]. Adding and subtracting both equations we get

$$0 = (\partial_0 + iqA_0 + im)\chi + (\vec{\partial} - iq\vec{A})\omega, \tag{1.14}$$

$$0 = (\partial_0 + iqA_0 - im)\omega - (\vec{\partial} - iq\vec{A})\chi. \tag{1.15}$$

7. This is why the first form of the Dirac wave, using α_j and β matrices which are indeed truly Hamiltonian, is another wave equation, which is not equivalent to the Dirac equation used here and in the whole relativistic part of the Standard Model. There, one goes from one equation to the other simply by multiplying on the left side by γ_0, yet forgetting to multiply also by γ_0^{-1} on the right side. It is thus a different wave equation. This may be seen, for instance, when the electromagnetic potential A does not commute with γ_0, which is the case if the magnetism of the electron or the weak interaction is not negligible.

The nonrelativistic approximation replaces (1.15) with another equation (De Broglie considered this as totally incorrect [57]), without its time derivative:

$$\omega = \frac{i}{2m}(\vec{\partial} - iq\vec{A})\chi, \tag{1.16}$$

and next by substituting ω in (1.14) we arrive at the Pauli equation:

$$(\partial_0 + iq\mathrm{A}_0 + im)\chi = \frac{1}{2im}(\vec{\partial} - iq\vec{\mathrm{A}})^2\chi. \tag{1.17}$$

The substitution of ω is justified by the Schrödinger equation where we have in modern notation $i\hbar c\partial_0\psi = E\psi$, and by the nonrelativistic approximation $E \approx m_0c^2$. Next we easily place the Pauli wave equation under the Hamiltonian form of the Schrödinger equation (1.1):

$$i\hbar c\partial_0\chi = H\chi;\ H\chi = (eA_0 + m_0c^2)\chi + \frac{\hbar^2}{2m_0}\Big(\vec{\partial} - i\frac{e}{\hbar c}\vec{A}\Big)^2\chi. \tag{1.18}$$

This is how quantum field theory (QFT) usually gets the "Hamiltonian of the Dirac equation." But the relativistic equation (1.2) is not used, only the approximation of this equation by the Pauli equation (1.17). Therefore here we will only use the Dirac equation and the Lagrangian formalism, never the approximation by the Pauli equation nor its Hamiltonian formalism [8], because we need to satisfy in their minutest details any implication of special relativity (SR). Further, it is to be noted that the replacement of the Dirac equation by the Pauli equation is not relativistic. This indeed takes away nothing from the results that QFT has so far obtained, just as general relativity (GR) does not suppress the results of Newtonian gravitation, which is only slightly corrected in low gravitational fields. The replacement by the Pauli wave works adequately so long as the quantum wave is reduced to only one spinor wave, mostly a left wave. But logically corrections must be made, so as to account for some "anomalies," when it is necessary to return to waves with both a left and a right part, or when two left waves are in use.

Now with the usual summation of upper and lower indices, we let:

$$\begin{gathered}\nabla := \sigma^\mu\partial_\mu;\ \widehat{\nabla} := \widehat{\sigma}^\mu\partial_\mu;\ A := \sigma^\mu A_\mu;\ \widehat{A} := \widehat{\sigma}^\mu A_\mu,\\ \mathbf{A} := \gamma^\mu A_\mu = \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix};\ \boldsymbol{\partial} := \gamma^\mu\partial_\mu = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix}.\end{gathered} \tag{1.19}$$

These calculations actually operate in Clifford algebra, more precisely in two algebras: the Pauli algebra which is also called Cl_3 (as Clifford algebra of $\mathbb{R}^3$), and the Clifford algebra of space-time $Cl_{1,3}$, which is the algebra

8. Moreover it is very difficult to obtain a Hamiltonian formalism in a truly relativistic fashion. This is the aim of I. Kanatchikov [80] as it is a prerequisite step for a unification of the different Hamiltonian formalisms of various parts of physical theory.

used more particularly by Hestenes, Boudet and Casanova [9]. A detailed introduction to Clifford algebra is presented in [22, 28, 36] and also in this book, Appendix A. We then detail in Appendix B the properties of $Cl_{1,3}$ which is isomorphic to a real subalgebra of the complex algebra $M_4(\mathbb{C})$. Actually $Cl_{1,3}$ is a left (and a right) modulus on the Cl_3 ring. In the calculations of the Dirac theory, this corresponds to the calculation by blocks of 2×2 matrices for the 4×4 matrices. The system (1.11), equivalent to the Dirac equation, is expressed as:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{i}{m} \begin{pmatrix} 0 & \nabla + iqA \\ \widehat{\nabla} + iq\widehat{A} & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{1.20}$$

This has the recursive functional form

$$\psi = f(\psi);\ f(\psi) = \frac{i}{m}\gamma^\mu(\partial_\mu + iqA_\mu)\psi = \frac{i}{m}(\boldsymbol{\partial} + iq\mathbf{A})\psi. \tag{1.21}$$

This form is very useful in studying the second-order equation that we come to now.

1.1.1 Second-order equation

By iteration of the functional f we get $\psi = f[f(\psi)]$ which means:

$$\begin{aligned} \psi &= \frac{i}{m}(\boldsymbol{\partial} + iq\mathbf{A})[\frac{i}{m}(\boldsymbol{\partial} + iq\mathbf{A})]\psi, \\ &= -\frac{1}{m^2}[\Box\psi + iq\boldsymbol{\partial}(\mathbf{A}\psi) + iq\mathbf{A}\boldsymbol{\partial}\psi - q^2\mathbf{A}^2\psi], \\ \Box :&= \boldsymbol{\partial}\boldsymbol{\partial} = \partial_0\partial_0 - \partial_1\partial_1 - \partial_2\partial_2 - \partial_3\partial_3. \end{aligned} \tag{1.22}$$

where $\Box$ is the D'Alembertian. Multiplying by m^2 this is equivalent to:

$$0 = (\Box + m^2 - q^2\mathbf{A}^2)\psi + iq[\boldsymbol{\partial}(\mathbf{A}\psi) + \mathbf{A}\boldsymbol{\partial}\psi]. \tag{1.23}$$

And we have:

$$\boldsymbol{\partial}(\mathbf{A}\psi) = (\boldsymbol{\partial}\mathbf{A})\psi + 2A^\mu\partial_\mu\psi - \mathbf{A}\boldsymbol{\partial}\psi; \tag{1.24}$$

$$\mathbf{F} = \begin{pmatrix} F & 0 \\ 0 & \widehat{F} \end{pmatrix} = \boldsymbol{\partial}\mathbf{A} = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix} = \begin{pmatrix} \nabla\widehat{A} & 0 \\ 0 & \widehat{\nabla}A \end{pmatrix}. \tag{1.25}$$

Then the electromagnetic field ($\mathbf{F}$ in space-time algebra, $F = \vec{E} + i\vec{H}$ in space algebra, where $\vec{E}$ is the electric field and $\vec{H}$ the magnetic field) allows us to obtain at the second order:

$$\begin{aligned} 0 &= (\Box + m^2 - q^2\mathbf{A}^2)\psi + iq[(\boldsymbol{\partial}\mathbf{A}) + 2A^\mu\partial_\mu - \mathbf{A}\boldsymbol{\partial} + \mathbf{A}\boldsymbol{\partial}]\psi, \\ 0 &= (\Box + m^2 - q^2\mathbf{A}^2)\psi + iq[\mathbf{F} + 2A^\mu\partial_\mu]\psi. \end{aligned} \tag{1.26}$$

We may remark that the classical electromagnetic field $\mathbf{F}$ comes with a field of operators $2A^\mu\partial_\mu$, in accordance with quantum field theory, where the electromagnetic field is a field of operators. We may also see two things that seem strange in this wave equation: firstly the field of operators is a scalar field, acting on ξ and η in the same way while the classical part $\mathbf{F}$ is a wholly bivector field, which is well established experimentally: this is linked to the complete absence of longitudinal polarization in light. Secondly the squares $m^2-q^2\mathbf{A}^2$ are of opposite signs, while the energy–momentum of the electron is the sum of a mechanical energy–momentum $m\mathbf{v}$ and an electromagnetic energy–momentum $q\mathbf{A}$ [9], instead of a difference between these two energy–momentum vectors. We thus replace (1.24) with an equality similar to the Leibniz rule for the derivative of a product:

$$\boldsymbol{\partial}(\mathbf{A}\psi) = \mathcal{F}(\psi) + \mathbf{A}\boldsymbol{\partial}\psi;\ \mathcal{F}(\psi) = \boldsymbol{\partial}(\mathbf{A}\psi) - \mathbf{A}\boldsymbol{\partial}\psi. \tag{1.27}$$

The second-order wave equation thus gives:

$$\begin{aligned} 0 &= (\Box + m^2 - q^2\mathbf{A}^2)\psi + iq[\mathcal{F}(\psi) + 2\mathbf{A}\boldsymbol{\partial}\psi] \\ &= (\Box + m^2 - q^2\mathbf{A}^2)\psi + iq[\mathcal{F}(\psi) - 2\mathbf{A}(iq\mathbf{A} + im)]\psi \\ &= [\Box + (m + q\mathbf{A})^2]\psi + iq\mathcal{F}(\psi). \end{aligned} \tag{1.28}$$

This yields both the expected sign for the energy–momentum term and an electromagnetic field that is actually a field of operators acting on ψ.

1.1.2 The form invariance of the Dirac equation

Attention please: it will be necessary to explain and correct a mistake made as early as the beginning of relativistic quantum physics, next used in all books on electron physics. Attention again: this form-invariance is very different from anything used in relativistic physics before quantum mechanics. First, space-time is considered in practice as a subset of the Cl_3 algebra (that was named the Pauli algebra) because, with Greek indices at 0, 1, 2, 3 and

$$\mathrm{x}^0 = ct\ ;\quad \vec{\mathrm{x}} = \mathrm{x}^1\sigma_1 + \mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3\ ;\quad \partial_\mu = \frac{\partial}{\partial \mathrm{x}^\mu}, \tag{1.29}$$

quantum physics as early as 1927 [10] wrote in the frame of Pauli's wave equation:

$$\vec{\mathrm{x}} = \begin{pmatrix} \mathrm{x}^3 & \mathrm{x}^1 - i\mathrm{x}^2 \\ \mathrm{x}^1 + i\mathrm{x}^2 & -\mathrm{x}^3 \end{pmatrix}. \tag{1.30}$$

9. This characterization of the energy–momentum of the electron as a sum is explicit in the work of Hestenes (formula 6.22c) in [76]).

10. All that must be known about Cl_3 is included in Appendix A, with a minimal level of mathematics. Thus an informed lecturer may skip this Appendix. Nevertheless a review might be useful.

This includes the whole physical space in the Cl_3 algebra. It is necessary because the $SU(2)$ group, which replaces the $SO(3)$ rotation group in the Pauli theory, is a subgroup of the Cl_3^* Lie group. And this was the starting point to extend the previous inclusion, by adding $\sigma_0 = I$ to the σ_j. This is linked to the representations of the Lorentz group [95][100] [11] :

$$\mathrm{x} = \mathrm{x}^\mu \sigma_\mu = \mathrm{x}^0 + \vec{\mathrm{x}} = \begin{pmatrix} \mathrm{x}^0 + \mathrm{x}^3 & \mathrm{x}^1 - i\mathrm{x}^2 \\ \mathrm{x}^1 + i\mathrm{x}^2 & \mathrm{x}^0 - \mathrm{x}^3 \end{pmatrix}. \tag{1.31}$$

And thus space-time is identified with the self-adjoint subset of the Pauli algebra Cl_3, which is the part of the M elements satisfying $M = M^\dagger$ (this identification is the starting point of Chapter 4).

The algebraic structure of Cl_3 is richer than the complex field. Instead of a single conjugation we now have three: the $P : M \mapsto \widehat{M}$ transformation is the main automorphism of this algebra. This P, called **parity** in quantum mechanics, allows us to separate even and odd parts and satisfies $\widehat{AB} = \widehat{A}\widehat{B}$. Next the antimorphism $M \mapsto \widetilde{M} = M^\dagger$ is the **reversion** which satisfies $\widetilde{AB} = \widetilde{B}\widetilde{A}$. These morphisms generate a third one, the product of the previous conjugations: $M \mapsto \overline{M} = \widehat{M}^\dagger = \mathrm{tr}(M) - M$ is an antimorphism since $\overline{AB} = \overline{B}\,\overline{A}$ (more details in A.3.3). And we get [12] :

$$\begin{aligned} &\widehat{\mathrm{x}} = \overline{\mathrm{x}} = \mathrm{x}^0 - \vec{\mathrm{x}}, & (1.32)\\ &||\mathrm{x}||^2 = \det(\mathrm{x}) = \mathrm{x}\widehat{\mathrm{x}} = \mathrm{x}\cdot\mathrm{x} = (\mathrm{x}^0)^2 - (\vec{\mathrm{x}})^2 = (\mathrm{x}^0)^2 - (\mathrm{x}^1)^2 - (\mathrm{x}^2)^2 - (\mathrm{x}^3)^2. \end{aligned}$$

The square $||\mathrm{x}||^2$ of the pseudo-norm of any space-time vector x is thus simply the determinant of this vector. Therefore the scalar product of two

11. After Rose's book [99] explaining with modern terms the results exposed in de Broglie's book [54], all succeeding reports on this invariance introduced the Lorentz transformation of the x^μ using first an infinitesimal Lorentz transformation. They actually only use the mathematical tool of the Lie algebra of the $SL(2,\mathbb{C})$ Lie group, which is the set $sl(2,\mathbb{C})$ of the 2×2 complex matrices with a null trace. And using a general result about the $SO(n)$ group, they suppose, without any proof, that this result is also true in the $SO(1,3)$ case! (They thus think that the Lie algebra of the $SO(1,3)$ Lie group is the same as the Lie algebra of the $SL(2,\mathbb{C})$ Lie group). The algebraic calculations are hence formally exact, but the strangeness of the situation is hidden into the exponential functions (notice the plural) which are applications of a single algebra onto two different groups. And these two different exponential functions are not presented as distinct, even if one of them applies to 2×2 complex Pauli matrices while the other applies to 4×4 real matrices. And there are two different groups, the group of 2×2 complex matrices with determinant 1 (where the unit is the unit matrix I_2) and the group of Lorentz transformations, where the unit is the identity transformation, id. It is nonsense to use both $\exp(0) = I_2$ and $\exp(0) = \mathrm{id}$, which should imply $I_2 = \mathrm{id}$. It is thus necessary to distinguish two exponential functions: $\exp_1$ applies the elements in the vicinity of 0 in $M_2(\mathbb{C})$ to $SL(2,\mathbb{C})$, while $\exp_2$ applies the same elements to the Lorentz group, where each element is one-to-one associated to a 4×4 real matrix. It may be correct that $\exp_1(0) = I_2$ and $\exp_2(0) = I_4$, and false that $I_2 = I_4$.

12. $\mathbb{R}$ is included in each Clifford algebra. Here this is equivalent to the identification between numbers a and scalar matrices aI_2. This simplifies many calculations.This identification is often used in mathematics; for instance the $\mathbb{R}$ field is put into the $\mathbb{C}$ field.

space-time vectors x and y reads:

$$\mathrm{x}\cdot\mathrm{y}=\frac{1}{2}(\mathrm{x}\widehat{\mathrm{y}}+\mathrm{y}\widehat{\mathrm{x}})=\frac{1}{2}(\widehat{\mathrm{x}}\mathrm{y}+\widehat{\mathrm{y}}\mathrm{x})=\mathrm{x}^0\mathrm{y}^0-\mathrm{x}^1\mathrm{y}^1-\mathrm{x}^2\mathrm{y}^2-\mathrm{x}^3\mathrm{y}^3. \quad (1.33)$$

The parity transformation $P : \mathrm{x} \mapsto \widehat{\mathrm{x}}$ is thus included in the geometric structure of space-time (see Chapter 4). Let M be any nonzero element in Cl_3 (that means any fixed nonzero Pauli matrix) and let R be the transformation of space-time into itself such that for any x is associated x' given by [13]:

$$\mathrm{x}'=\mathrm{x}'^0+\vec{\mathrm{x}}\,'=R(\mathrm{x})=M\mathrm{x}M^\dagger. \quad (1.34)$$

We note, if $\det(M)\neq 0$:

$$\det(M)=re^{i\theta}\ ,\quad r=|\det(M)|\ ;\quad \underline{M}:=r^{-1/2}M. \quad (1.35)$$

Thus r is the modulus and θ is an argument of the determinant of M. Attention, please: r is not the determinant of M, it is only the modulus of the determinant. We get:

$$\begin{aligned}(\mathrm{x}'^0)^2-(\mathrm{x}'^1)^2-(\mathrm{x}'^2)^2-(\mathrm{x}'^3)^2&=\det(\mathrm{x}')=\det(M\mathrm{x}M^\dagger)\\ &=re^{i\theta}\det(\mathrm{x})re^{-i\theta}=r^2[(\mathrm{x}^0)^2-(\mathrm{x}^1)^2-(\mathrm{x}^2)^2-(\mathrm{x}^3)^2]. \quad (1.36)\end{aligned}$$

Therefore R multiplies any space-time distance by r and we call this transformation "similitude with ratio r." We refer to M as the dilator of the similitude R. Even if, since 1928, most physicists confused similitude and dilator, here we will use two distinct words because **a similitude is not a dilator**. We now consider the $\underline{R}$ transformation such that:

$$\underline{\mathrm{x}}'=\underline{R}(\mathrm{x}):=\underline{M}\mathrm{x}\underline{M}^\dagger. \quad (1.37)$$

$$\mathrm{x}'=r^{1/2}\underline{M}\mathrm{x}r^{1/2}\underline{M}^\dagger=r\underline{M}\mathrm{x}\underline{M}^\dagger=r\underline{R}(\mathrm{x});\ R=r\underline{R}. \quad (1.38)$$

Therefore R is the product, in any order, of $\underline{R}$ and of a homothety with ratio r. And since we defined $\underline{M}$ such that:

$$|\det(\underline{M})|=1, \quad (1.39)$$

the set of the $\underline{M}$ is the Lie group $\mathcal{G}$ of the elements in $GL(2,\mathbb{C})$ such that $|\det(\underline{M})|=1$. We then have in place of (1.36):

$$\begin{aligned}(\underline{\mathrm{x}}'^0)^2-(\underline{\mathrm{x}}'^1)^2-(\underline{\mathrm{x}}'^2)^2-(\underline{\mathrm{x}}'^3)^2&=\det(\underline{\mathrm{x}}')=\det(\underline{M}\mathrm{x}\underline{M}^\dagger)\\ &=|\det(M)|^2\det(\mathrm{x})=(\mathrm{x}^0)^2-(\mathrm{x}^1)^2-(\mathrm{x}^2)^2-(\mathrm{x}^3)^2. \quad (1.40)\end{aligned}$$

Thus $\underline{R}$ is a Lorentz transformation. With the usual summation convention of upper and lower indices, we let:

$$\mathrm{x}'^\mu=R^\mu_\nu\mathrm{x}^\nu;\ \underline{\mathrm{x}}'^\mu=\underline{R}^\mu_\nu\mathrm{x}^\nu. \quad (1.41)$$

13. Only one other possibility exists: $\vec{\mathrm{x}}\,'=R(\mathrm{x})=M\mathrm{x}M^\dagger/\sqrt{r}$. With $N:=\sqrt[4]{r}M$ we have $\vec{\mathrm{x}}\,'=R(\mathrm{x})=N\mathrm{x}N^\dagger$, and we recover our simplest form.

(R^μ_ν) is the real 4×4 matrix of the R similitude and $(\underline{R}^\mu_\nu)$ is the real 4×4 matrix of the Lorentz transformation $\underline{R}$. We hence have the following (see A.4.2) for any dilator $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$:

$$2R^0_0 = |a|^2 + |b|^2 + |c|^2 + |d|^2 > 0, \tag{1.42}$$

and x'^0 then has the same sign as x^0 at the origin: the similitude R, and hence also $\underline{R}$, conserves the arrow of time. Moreover, for any dilator M in Cl_3, and even if the ratio is null, we obtain (detailed calculations in A.4.5) the following simple and nontrivial equality:

$$\det(R^\mu_\nu) = r^4. \tag{1.43}$$

If r is nonzero then $r^4 > 0$: $\det(R) > 0$. Thus R conserves the orientation of space-time, and since the transformation conserves the orientation of time, R also conserves the orientation of space. Moreover we get:

$$\det(\underline{R}^\mu_\nu) = 1. \tag{1.44}$$

This concludes the demonstration that $\underline{R}$ is a transformation in the Lorentz group (the transformation group conserving the space-time metric). But this group is not the invariance group of the wave with spin 1/2. Only the restricted Lorentz group is obtained from (1.34). The true Lorentz group needs the use of the P and T transformations. And neither P nor T is an exact symmetry of nature; both are violated in weak interactions. Therefore we will only use the restricted Lorentz group. We consider again the f function which associates to the dilator M the similitude $R = f(M)$. Let M' be any other dilator, with:

$$\det(M') = r'e^{i\theta'} \quad ; \quad R' = f(M') \quad ; \quad \mathrm{x}'' = M'\mathrm{x}'M'^\dagger. \tag{1.45}$$

We then get :

$$\begin{aligned} \mathrm{x}'' = M'\mathrm{x}'M'^\dagger &= M'(M\mathrm{x}M^\dagger)M'^\dagger = (M'M)\mathrm{x}(M'M)^\dagger \\ R' \circ R &= f(M') \circ f(M) = f(M'M), \end{aligned} \tag{1.46}$$

and with $r \neq 0$, f becomes a homomorphism [14] from the $(Cl_3^*, \times)$ group into the $(D^*, \circ)$ group where D^* is the set of all similitudes with nonzero ratio. These two groups are Lie groups: $(Cl_3^*, \times)$ is the 8-dimensional $GL(2, \mathbb{C})$ group. But $(D^*, \circ)$ is only a 7-dimensional Lie group: one dimension disappears because the kernel of f is not reduced to the neutral element. Let θ be any real number and let M be a dilator such that:

$$M = e^{i\theta/2} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}; \ \det(M) = e^{i\theta}, \tag{1.47}$$

14. Most quantum physicists use the name “representation” in place of homomorphism.

we then get:

$$\mathrm{x}' = M\mathrm{x}M^\dagger = e^{i\theta/2}\mathrm{x}e^{-i\theta/2} = \mathrm{x}. \tag{1.48}$$

$f(M)$ is thus the neutral element and M belongs to the kernel of f. Therefore the kernel is a one-parameter group and only seven parameters remain in D^*. Six of them define a proper Lorentz transformation and the seventh is the ratio of the similitude r. For instance if the dilator is

$$M = e^{a+b\sigma_1} = e^a[\cosh(b) + \sinh(b)\sigma_1], \tag{1.49}$$

thus the similitude R defined in (1.34) satisfies

$$\begin{aligned}\mathrm{x}' = M\mathrm{x}M^\dagger &= e^{a+b\sigma_1}(\mathrm{x}^0 + \mathrm{x}^1\sigma_1 + \mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3)e^{a+b\sigma_1}\\ &= e^{2a}[e^{2b\sigma_1}(\mathrm{x}^0 + \mathrm{x}^1\sigma_1) + \mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3].\end{aligned} \tag{1.50}$$

We hence get:

$$\begin{aligned}\mathrm{x}'^0 + \mathrm{x}'^1\sigma_1 &= e^{2a}\big[\cosh(2b) + \sinh(2b)\sigma_1\big](\mathrm{x}^0 + \mathrm{x}^1\sigma_1),\\ \mathrm{x}'^0 &= e^{2a}\big[\cosh(2b)\mathrm{x}^0 + \sinh(2b)\mathrm{x}^1\big];\ \mathrm{x}'^2 = e^{2a}\mathrm{x}^2,\\ \mathrm{x}'^1 &= e^{2a}\big[\sinh(2b)\mathrm{x}^0 + \cosh(2b)\mathrm{x}^1\big];\ \mathrm{x}'^3 = e^{2a}\mathrm{x}^3.\end{aligned} \tag{1.51}$$

We can see that the similitude R is the product, in any order, of a proper Lorentz transformation (boost) mixing the temporal component x^0 and the spatial component x^1 by a homothety with ratio $r = e^{2a}$. Next if:

$$M = e^{a+bi\sigma_1} = e^a[\cos(b) + \sin(b)i\sigma_1], \tag{1.52}$$

then the similitude R defined in (1.34) satisfies

$$\begin{aligned}\mathrm{x}' = M\mathrm{x}M^\dagger &= e^{a+bi\sigma_1}(\mathrm{x}^0 + \mathrm{x}^1\sigma_1 + \mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3)e^{a-bi\sigma_1}\\ &=e^{2a}[\mathrm{x}^0 + \mathrm{x}^1\sigma_1 + e^{2bi\sigma_1}(\mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3)].\end{aligned} \tag{1.53}$$

We thus have:

$$\begin{aligned}\mathrm{x}'^2\sigma_2 + \mathrm{x}'^3\sigma_3 &= e^{2a}\big[\cos(2b) + \sin(2b)i\sigma_1\big](\mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3),\\ \mathrm{x}'^2 &= e^{2a}\big[\cos(2b)\mathrm{x}^2 + \sin(2b)\mathrm{x}^3\big];\ \mathrm{x}'^0 = e^{2a}\mathrm{x}^0,\\ \mathrm{x}'^3 &= e^{2a}\big[-\sin(2b)\mathrm{x}^2 + \cos(2b)\mathrm{x}^3\big];\ \mathrm{x}'^1 = e^{2a}\mathrm{x}^1.\end{aligned} \tag{1.54}$$

And so R is the product of a rotation with axis $O\mathrm{x}^1$ and a $2b$ angle by the same homothety with ratio $r = e^{2a}$. Considering the distinction between the dilator M and the similitude R is absolutely necessary. This distinction was unfortunately never made prior to our work: the Dirac theory confused M and R, so much so that the same name was given to these different objects! Here we will absolutely avoid calling M a Lorentz transformation since it is a very different object, even if each dilator M allows us to define a similitude. The Lie group of the dilators, $Cl_3^* = GL(2,\mathbb{C})$, and the group $\mathcal{D}^*$ of the similitudes are very different. They do not have the same topology, nor do they even have the same dimension. Thus they do not have the same set of infinitesimal elements that is the same Lie algebra. Hence they must not be confused, even in the neighborhood of the neutral element. The previous calculations are simple because we start from the dilator M to get the similitude R. The reverse calculation is impossible and it is actually nonsense, because the similitude R is the image of the dilator M by the function f, and this function is not invertible. No isomorphism can exist between the 8-dimensional group of the dilators and the 7-dimensional group of the similitudes.

Restricted Lorentz group

If we now add the condition $|\det(M)| = 1$, we identify M and $\underline{M}$. The set of the dilators $\underline{M}$ is $\mathcal{G}$, and (1.36) is reduced to:

$$(\mathrm{x}'^0)^2 - (\mathrm{x}'^1)^2 - (\mathrm{x}'^2)^2 - (\mathrm{x}'^3)^2 = (\mathrm{x}^0)^2 - (\mathrm{x}^1)^2 - (\mathrm{x}^2)^2 - (\mathrm{x}^3)^2. \quad (1.55)$$

The similitude R is then simply a Lorentz transformation and the set of R is called the restricted Lorentz group, usually denoted $\mathcal{L}_+^\uparrow$. The time orientation and space orientation are separately conserved. The determinant satisfies:

$$1 = |e^{i\theta}| \;\; ; \;\; \theta \in \mathbb{R}. \quad (1.56)$$

The 7-dimensional Lie algebra of $\mathcal{G}$ and the 6-dimensional Lie algebra of $\mathcal{L}_+^\uparrow$ cannot be confused. The first one is the Lie algebra of a 7-dimensional real Lie group, while the second is the Lie algebra of a 6-dimensional real Lie group. What happens is not only that dilation and similitude are mixed up, another mistake is to confuse $\det(M) = 1$ (which defines $SL(2,\mathbb{C})$) with $|\det(M)| = 1$ (which defines $\mathcal{G}$). The reason of these mistakes is easy to understand: the Pauli wave equation induced a mixing up of the $SO(3)$ and $SU(2)$ Lie groups, which have the same Lie algebra: the algebra $su(2)$ of the Hermitian matrix with a null trace. Since $\det[\exp(M)] = \exp(\mathrm{tr}(M))$, the element $\exp(M), M \in su(2)$ has a determinant 1.

The exponential function is general in Lie group theory. It is a function from a neighborhood (which may be small) of the zero in the Lie algebra, on a neighborhood of the unity in the Lie group. In the simple case of the $GL(n,\mathbb{C})$ Lie group, which has the algebra of $n \times n$ complex matrices as Lie algebra, the exponential function is simply:

$$\exp(M) = e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}. \quad (1.57)$$

Nonrelativistic quantum mechanics uses two simple properties of $SU(2)$: first, any element M in $SU(2)$ satisfies:

$$M = \exp(ia^j\sigma_j), \;\; j = 1,2,3, \;\; a^j \in \mathbb{R}. \quad (1.58)$$

Second, for any rotation R in $SO(3)$, a M exists, defined up a sign, such that $R = f(M)$ where f is the homomorphism applying each dilator on the associated similitude. We now consider $M = -1 + \sigma_1 + i\sigma_2$, which is an element of $SL(2,\mathbb{C})$ since $\det(M) = 1$. And M only satisfies:

$$\begin{aligned}(\sigma_1 + i\sigma_2)^2 = 0;\ \exp(\sigma_1 + i\sigma_2) = 1 + (\sigma_1 + i\sigma_2) + 0,\\ M = -[1 - (\sigma_1 + i\sigma_2)];\ M = -\exp[-(\sigma_1 + i\sigma_2)],\\ M = \exp(i\pi - \sigma_1 - i\sigma_2).\end{aligned} \tag{1.59}$$

Hence the exponential function, in the $SL(2,\mathbb{C})$ case has properties different from the exponential in the $SU(2)$ case. The lack of understanding of this difference induced false theorems: Bacry [1] claimed (without proof!) that any Lorentz transformation in $\mathcal{L}_+^\uparrow$ is the product LR of a boost L and a rotation R, while Naïmark [95] proved that any Lorentz transformation in $\mathcal{L}_+^\uparrow$ reads ub_1v where u and v are rotations and b_1 is a one-parameter boost. This also implies $7 = 3 + 1 + 3$ parameters. And it is the restriction $\underline{f}$ of f to the $\mathcal{G}$ group which an homomorphism from $\mathcal{G}$ into $\mathcal{L}_+^\uparrow$, and with the same kernel, the 1-dimensional $U(1)$ group, thus $\underline{f}$ is not invertible : the calculation of M from R is impossible.

Nevertheless the $\mathcal{G}$ group contains as a subgroup the $SU(2)$ group of the 2×2 unitary matrices with determinant 1. The restriction of $\underline{f}$ to this subgroup is a homomorphism from $SU(2)$ into the $SO(3)$ rotation group in space. The kernel of this homomorphism is now reduced to $\{\pm 1\}$. This is the basis of all calculations using the spin of a system of electrons. Of course all results of these calculations, like the 6j and 9j symbols, are exact since they properly use theorems on Lie groups, and they are calculated not by composing rotations but by actually multiplying unitary matrices.

1.2 Extended invariance

The first important change that we now propose is the removal of the condition $|\det(M)| = 1$ and its replacement with $\det(M) \neq 0$ (this condition is used only to obtain the structure of multiplicative group). That is to say, we replace the 7-dimensional $\mathcal{G}$ Lie group by the 8-dimensional $GL(2,\mathbb{C}) = Cl_3^*$ Lie group itself. This group is also the multiplicative group in Cl_3, and Cl_3 is the Lie algebra of its subset Cl_3^*. We may put forward four reasons:

1. This is possible (and very surprising) because the properties (1.36), (1.42) and (1.43) are general and do not suppose that $\det(M) = 1$ nor $|\det(M)| = 1$ [18, 20, 21]. Nowhere do these restrictive equalities seem necessary for the wave of the electron. To see this it is enough to never use infinitesimal transformations, contrary to most course books, and simply to directly calculate in the Lie groups.

2. This value of the determinant of the dilator has no geometric origin in space-time, while gravitation is linked to the geometry of space-time. And

Cl_3^* is obviously a geometric group since it is the multiplicative group of the algebra including space-time.

3. The Russian physicist V. Fock [71] rebuilt general relativity from the properties of electromagnetism and gravitation. His starting point was, as it was for Einstein, the invariance of the speed of light regarding any frame of reference, even in movement of translation. Since light is an electromagnetic wave, Fock considered an electromagnetic wavefront. He then proved that the transformation linking the coordinates of an event was necessarily linear, and he afterwards proved that the R transformation was necessarily a Lorentz similitude, which means the product of a Lorentz transformation and a homothety. But he was working from electromagnetism (thus solely from the similitude R) and had no luck to introduce the dilator M that comes only with the quantum wave. Of course Fock was a master of quantum mechanics but he no more accounted for the difference between M and R than did other physicists. Since he wanted to get only the Lorentz transformations he claimed that the ratio of the homothety was necessarily 1. Even though the invariance group of electromagnetic laws was known to be much larger than the Lorentz group and included the similitudes, Fock's error flooded all the Russian work in this domain of physics. Afterwards the success of Landau's books extended this error to the whole of QFT.

4. This extended invariance will allow us to understand the geometry of the four kinds of interactions in physics (electromagnetism, weak interactions, strong interactions and gravitation), the quantization of kinetic momentum, the proper nature of the electromagnetic field, and more in the next chapters. The power of this approach comes from the inclusion of parity transformation in the geometry of space-time resulting from $||\mathrm{x}||^2 = \mathrm{x}P(\mathrm{x})$.

We now return to the Dirac equation and we look at how the wave with spin 1/2 comes to be, without imposing the condition $\det(M) = 1$. First the right wave ξ and the left wave η do not transform similarly:

$$\xi' = \xi'(\mathrm{x}') = M\xi = M\xi(\mathrm{x});\ \eta' = \eta'(\mathrm{x}') = \widehat{M}\eta = \widehat{M}\eta(\mathrm{x}). \tag{1.60}$$

This is actually the origin of the existence of right waves and left waves: they do not transform similarly in Lorentz transformations. The change is caused by the boost (1.49) because we have

$$\xi' = e^{a+b\sigma_1}\xi;\ \eta' = e^{a-b\sigma_1}\eta. \tag{1.61}$$

With transformations like (1.52), which are rotations, we have $\widehat{M} = M$, and so the right wave and left wave transform similarly. Consequently the theory of weak interactions may only start from the Dirac wave equation, which is relativistic, the only one able to distinguish between right waves and left waves (in contrast to the Pauli theory which only knows left-handed and right-handed parts of the wave). This is well known in the Standard Model. To see how the system (1.11) is changed (a system equivalent to the

Dirac equation), we need the following nontrivial relation (details in A.4.4):

$$\nabla = \overline{M}\nabla'\widehat{M};\ \nabla' = \sigma^\mu \partial'_\mu;\ \partial'_\mu = \frac{\partial}{\partial \mathrm{x}'^\mu}. \tag{1.62}$$

Hence we must separate the space-time vectors transforming like x which we call contravariant vectors, from the vectors transforming like ∇ which we call covariant vectors. The two supplementary dimensions of the invariance group induce new constraints which are added to the constraints of relativistic invariance: in tensor calculus we now have no possibility of moving a tensor index up or down. We also cannot replace a contravariant n vector with the covariant ∇ like Lasenby did in [83]. The previous constraints remain: for the transformations of the kind (1.49) as well as for those of the kind (1.52), when we have a θ angle with the transformation of ξ and η, we get a double angle 2θ with the transformation of x and ∇. This too is well known in quantum physics. The system (1.11) becomes, if qA is transformed like ∇ (which is necessary for the gauge invariance):

$$\begin{aligned}\xi' &= M\xi = M\frac{i}{m}(\overline{M}\nabla'\widehat{M} + iq'\overline{M}A'\widehat{M})\eta = M\overline{M}\frac{i}{m}(\nabla' + iq'A')\eta',\\ \eta' &= \widehat{M}\eta = \widehat{M}\frac{i}{m}(\widetilde{M}\widehat{\nabla}'M + iq'\widetilde{M}\widehat{A}'M)\xi = \widehat{M}\widetilde{M}\frac{i}{m}(\widehat{\nabla}' + iq'\widehat{A}')\xi'.\end{aligned} \tag{1.63}$$

And with $\det(M) = re^{i\theta}$ we have:

$$M\overline{M} = re^{i\theta};\ \widehat{M}\widetilde{M} = re^{-i\theta};\ \overline{M} = re^{i\theta}M^{-1}. \tag{1.64}$$

The system (1.63) can hence be expressed as:

$$\xi' = re^{i\theta}\frac{i}{m}(\nabla' + iq'A')\eta';\ \eta' = re^{-i\theta}\frac{i}{m}(\widehat{\nabla}' + iq'\widehat{A}')\xi'. \tag{1.65}$$

In the particular case where M belongs to $SL(2,\mathbb{C})$, this is reduced to

$$\xi' = \frac{i}{m}(\nabla' + iqA')\eta';\ \eta' = \frac{i}{m}(\widehat{\nabla}' + iq\widehat{A}')\xi'. \tag{1.66}$$

So we are right in saying that the form of the wave equation is unchanged [15] .

For a complete use of the extended invariance group we shall first change the appearance of the wave equation, placing any calculation in the framework of the same algebra. This means that all elements of the wave equation – differential operators, potentials, addition and multiplication, values of the wave, space and time – will be put into the same algebraic-geometrical structure. Next we shall change the wave equation itself by simplifying the Lagrangian density from whence the equation comes.

15. A lecturer in a hurry may not see what differs here from the conventional exposition of quantum mechanics. In fact, there is no difference except for the distinction between ∇ acting on η and $\widehat{\nabla}$ acting on ξ, and the distinction between dilator and similitude. But these distinctions (linked to parity) will prove essential for geometric properties of the space-time manifold in Chapter 4.

1.3 The Dirac equation in Cl_3

It is possible to use instead the initial formalism of Dirac matrices (a 16-dimensional linear space on $\mathbb{C}$, and a 32-dimensional linear space on $\mathbb{R}$) $Cl_{1,3}$ (a 16-dimensional linear space on $\mathbb{R}$) called space-time algebra. This was done by Hestenes [74]–[78]. Here we present another formalism still less costly in dimensions, using only the Cl_3 algebra, a 8-dimensional linear space on $\mathbb{R}$.[16] The form used as early as 1928 for the relativistic invariance that we just studied is the first reason for our choice. This invariance uses only Cl_3^*, the multiplicative group of Cl_3. Second, the Dirac wave of the electron has value only in Cl_3, not in the full space-time algebra. Third, which is the main reason, the use of Cl_3 was discovered by one of the present authors as sufficient for the description of the entire Dirac theory [15, 16]. It will allow us to obtain the link between left and right spinors in the simplest way.

For the expression of the Dirac wave in Cl_3, it is enough to replace the column matrices ξ and η by 2×2 matrices with a null column. This changes nothing concerning the calculation because the product of matrices is a row-to-column multiplication which operates separately on each row of the left matrix and on each column of the right matrix in any product. For an easier calculation of tensor densities we include a $\sqrt{2}$ factor and let:

$$R^1 := \sqrt{2}\begin{pmatrix}\xi_1^1 & 0\\ \xi_2^1 & 0\end{pmatrix};\ \widehat{L}^1 := \sqrt{2}\begin{pmatrix}\eta_1^1 & 0\\ \eta_2^1 & 0\end{pmatrix},$$
$$\phi := R^1 + L^1 = \sqrt{2}(\xi^1 \ \ \widehat{\eta}^1) = \sqrt{2}\begin{pmatrix}\xi_1^1 & -\eta_2^{1*}\\ \xi_2^1 & \eta_1^{1*}\end{pmatrix}. \tag{1.67}$$

We note that the complex conjugate of z is either $\overline{z}$, which is the usual notation in mathematics, or z^*, which is usual in the Dirac theory. We have:

$$\widehat{\phi} = \sqrt{2}\begin{pmatrix}\eta_1^1 & -\xi_2^{1*}\\ \eta_2^1 & \xi_1^{1*}\end{pmatrix} = \sqrt{2}\begin{pmatrix}\eta^1 & \widehat{\xi}^1\end{pmatrix},$$
$$R^1 = \phi\frac{1+\sigma_3}{2} = \sqrt{2}(\xi^1 \ \ 0);\ \widehat{L}^1 = \widehat{\phi}\frac{1+\sigma_3}{2} = \sqrt{2}(\eta^1 \ \ 0). \tag{1.68}$$

The link between ϕ, R^1 and L^1 is independent of the reference frame used because transformations in (1.60) are equivalent[17] to:

$$\phi' = \phi'(x') = M\phi = M\phi(x);\ \widehat{\phi}' = \widehat{\phi}'(x') = \widehat{M}\widehat{\phi} = \widehat{M}\widehat{\phi}(x). \tag{1.69}$$

16. The use of Cl_3 was first promoted by W. Baylis [3].

17. This equivalence is not trivial and comes from the fact that σ_2 is imaginary while σ_1 and σ_3 are real matrices. The result is, for any ϕ, that the P transformation $M \mapsto \widehat{M}$ exchanges ξ and η.

The system (1.11) (equivalent to the Dirac equation) is then expressed as:

$$0 = (\nabla + iqA)\widehat{\phi}\frac{1+\sigma_3}{2} + im\phi\frac{1+\sigma_3}{2},$$
$$0 = (\widehat{\nabla} + iq\widehat{A})\phi\frac{1+\sigma_3}{2} + im\widehat{\phi}\frac{1+\sigma_3}{2}. \tag{1.70}$$

Applying $P : M \mapsto \widehat{M}$ [18] to the second equation, we get the equivalent system:

$$0 = (\nabla + iqA)\widehat{\phi}\frac{1+\sigma_3}{2} + im\phi\frac{1+\sigma_3}{2},$$
$$0 = (\nabla - iqA)\widehat{\phi}\frac{1-\sigma_3}{2} - im\phi\frac{1-\sigma_3}{2}. \tag{1.71}$$

This system is itself equivalent to the single equation via addition:

$$0 = \nabla\widehat{\phi} + qA\widehat{\phi}i\sigma_3 + m\phi i\sigma_3, \tag{1.72}$$

because each line of the system (1.71) is obtained by applying on (1.72) a projector on the right and left wave, that is, in the Pauli algebra the multiplication on the right side by $(1 \pm \sigma_3)/2$. We will finish the simplification of the Dirac equation by multiplying the right side with $-i\sigma_3 = \sigma_{21} = \sigma_2\sigma_1$. The Dirac equation is hence equivalent to:

$$0 = \nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + m\phi, \tag{1.73}$$

and, using the parity transformation P, is equivalent to:

$$0 = \widehat{\nabla}\phi\sigma_{21} + q\widehat{A}\phi + m\widehat{\phi}. \tag{1.74}$$

Once again and despite the very different look **these equations are exactly the Dirac equation**. The gauge invariance now has the form:

$$\phi \mapsto \phi' = \phi e^{ia\sigma_3} \ ; \quad A \mapsto A' = A - \frac{1}{q}\nabla a \tag{1.75}$$

When quantum physics becomes relativistic, the multiplication by the non-specific imaginary number i must then be replaced by the multiplication on the right side by $i_3 = i\sigma_3$. This term is, from the geometrical point of view, a bivector or 2-vector, which means an oriented area (a cross product). It is thus different from the 3-vector $i = \sigma_1\sigma_2\sigma_3$ which is an oriented volume. This other i is the one used for instance in the expression for the electromagnetic field as a sum of an electric field and a magnetic field: $F = \vec{E} + i\vec{H}$.

18. P is, from the mathematical point of view, the main automorphism in Cl_3 changing i into $-i$ and σ_3 into $-\sigma_3$. From the physical point of view P stands for "parity" which exchanges right and left waves.

For anyone used to a single i, this is a major change, yet absolutely necessary from the geometric point of view: a 2-vector is an oriented area, while a 3-vector is an oriented volume, and an area is not a volume.

All objects present in the wave equation are now in the same algebra. Calculations using 2×2 matrices are much simpler than using 4×4 Dirac matrices. Moreover we get supplementary properties linked to the minimal dimension of the 2×2 matrices, such as the fact that comatrices giving an inverse matrix are reduced to numbers.

1.3.1 Plane wave

This section uses the simplest case where the interaction with the exterior electromagnetic field is negligible [19] . We then let $A = 0$. The Dirac equation in the Cl_3 algebra is thus reduced to

$$\nabla\widehat{\phi}\sigma_{21} + m\phi = 0. \tag{1.76}$$

We consider a plane wave with a phase φ such that:

$$\phi = \phi_0 e^{\varphi\sigma_{21}} \ ; \quad \varphi = m\mathrm{v}_\mu \mathrm{x}^\mu. \tag{1.77}$$

We use the space-time vector called the reduced velocity:

$$\mathrm{v} = \sigma^\mu \mathrm{v}_\mu, \tag{1.78}$$

and ϕ_0 is a fixed term which gives

$$\nabla\widehat{\phi}\sigma_{21} = \sigma^\mu \partial_\mu(\widehat{\phi}_0 e^{\varphi\sigma_{21}})\sigma_{21} = -m\mathrm{v}\widehat{\phi}. \tag{1.79}$$

Therefore the Dirac equation is equivalent to

$$\phi = \mathrm{v}\widehat{\phi}. \tag{1.80}$$

By using the P conjugation this is equivalent to

$$\widehat{\phi} = \widehat{\mathrm{v}}\phi. \tag{1.81}$$

Then combining the two previous equalities we have:

$$\phi = \mathrm{v}(\widehat{\mathrm{v}}\phi) = (\mathrm{v}\widehat{\mathrm{v}})\phi = (\mathrm{v} \cdot \mathrm{v})\phi. \tag{1.82}$$

If ϕ is invertible we must then get:

$$1 = \mathrm{v} \cdot \mathrm{v} = \mathrm{v}_0^2 - \vec{\mathrm{v}}^{\,2}, \tag{1.83}$$

$$\mathrm{v}_0^2 = 1 + \vec{\mathrm{v}}^{\,2} \ ; \quad \mathrm{v}_0 = \pm\sqrt{1 + \vec{\mathrm{v}}^{\,2}}, \tag{1.84}$$

19. We may be doubtful of the possibility of ignoring the charge of the electron. The plane wave solutions are actually presented only in the interest of easier calculation. Yet later, Fourier analysis allows us to extend the utility of this calculation much further.

with *a priori* two possibilities for the sign. The minus sign implied a negative energy for the particle – this was at the beginning a serious dissatisfaction for Dirac. He hoped to get rid of the negative quantities, supposedly nonphysical, that came from the Klein-Gordon equation which was the relativistic version of the Schrödinger equation. And it was impossible to suppress these negative energies [20] . They were necessary for obtaining any wave as a sum of plane waves from the Fourier transformation, or for getting a small enough wave packet. Six years later the discovery of the positron, a particle which had the same mass as the electron yet an opposite charge, completely changed the problem: these plane waves with negative energies were associated to the positron. And this association was considered as the triumph of the Dirac theory. Nevertheless these waves with negative energies induced formidable problems when their effects on the emission or absorption of light were calculated. Moreover positrons seemed to have the same proper mass, not a mass opposite to the proper mass of the electron (we will see this later).

The calculation that we present here is **much simpler** than the calculation made in relativistic quantum physics books using complex 4×4 matrices. This is a sufficient reason, among many others, to prefer the Cl_3 algebra to the Dirac algebra.

1.3.2 Tensor densities without a derivative

The $\mathrm{J} = \mathrm{J}^\mu \sigma_\mu$ current is one of the tensor quantities of the Dirac theory such that the definition $\mathrm{J}^\mu = \overline{\psi}\gamma^\mu\psi$ of the four components is made from the spinor wave without a partial derivative. We may first remark with L. de Broglie [54] about the strange character of these tensor densities which had no true equivalent in physics before quantum theory. Several other similar quantities were quickly noted [54], first a scalar one:

$$\Omega_1 = \overline{\psi}\psi \ ; \quad \overline{\psi} = \psi^\dagger\gamma_0 = (\eta^{1\dagger} \ \xi^{1\dagger}), \tag{1.85}$$

where $M^\dagger$ is the adjoint matrix (transposed conjugate). Next the six:

$$S^{\mu\nu} = i\overline{\psi}\gamma^\mu\gamma^\nu\psi, \tag{1.86}$$

are considered as the components of an antisymmetric tensor of rank two. The four K^μ:

$$\mathrm{K}^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi \ ; \quad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{1.87}$$

20. Plane waves, even if they are calculated here much more simply, are not the panacea often presented. De Broglie warned us against the abuse made with these waves: a wave unlimited in space or in time does not exist in nature. In an electronic microscope a train of waves is always limited in space and in time. We saw in 1.2 how Fock made use of an electromagnetic wavefront. Moreover this calculation neglects the charge term, as if we were able to remove or restore a charge at will. Hence plane waves are much too virtual and unreal to be very interesting from the physical point of view.

are considered the components of a pseudovector in space-time, theoretically linked to an antisymmetric tensor of rank three, even if this link is never used. Finally:

$$\Omega_2 = -i\overline{\psi}\gamma_5\psi, \tag{1.88}$$

is a relativistic invariant and allows us to define the main invariant ρ and the Yvon-Takabayasi β angle:

$$\Omega_1 = \rho\cos\beta \ ; \quad \Omega_2 = \rho\sin\beta \ ; \quad \Omega_1 + i\Omega_2 = \rho e^{i\beta}. \tag{1.89}$$

With the Weyl spinors (left and right waves) we get:

$$\begin{aligned} \Omega_1 &= \xi^{1\dagger}\eta^1 + \eta^{1\dagger}\xi^1 \ ; \quad \Omega_2 = i(\xi^{1\dagger}\eta^1 - \eta^{1\dagger}\xi^1), \\ \rho e^{i\beta} &= \Omega_1 + i\Omega_2 = 2\eta^{1\dagger}\xi^1 = 2(\eta_1^{1*}\xi_1^1 + \eta_2^{1*}\xi_2^1), \\ \rho e^{-i\beta} &= \Omega_1 - i\Omega_2 = 2\xi^{1\dagger}\eta^1 = 2(\eta_1^1\xi_1^{1*} + \eta_2^1\xi_2^{1*}). \end{aligned} \tag{1.90}$$

These tensor densities were intensively studied because physicists were very eager to link these quantities to classical physics, where all studied quantities are vectors and tensors (but they are not tensor densities). Actually these 16 tensor densities that we previously detailed know nothing of the phase of the wave: they contain the product of $\overline{\psi} = \psi^\dagger\gamma_0$ by ψ and are then gauge-invariant under the electric gauge [21] . Thus we cannot substitute the dynamics of these densities for the dynamics of the ψ wave itself. Yet within the Cl_3 framework, Ω_1 and Ω_2 are very simple:

$$\det(\phi) = \phi\overline{\phi} = \overline{\phi}\phi = \Omega_1 + i\Omega_2 = \rho e^{i\beta}. \tag{1.91}$$

So ρ is the modulus and the Yvon-Takabayasi β is an argument of the determinant of $\phi = \phi(\mathrm{x})$; hence they depend on x. Moreover, ϕ is invertible if and only if $\rho \neq 0$. The detailed calculation of J_μ and K_μ (see A.4.2) using ξ^1 and η^1 gives

$$\mathrm{J} = \mathrm{J}^\mu\sigma_\mu = \phi\sigma_0\phi^\dagger = \phi\phi^\dagger \ ; \quad \mathrm{K} = \mathrm{K}^\mu\sigma_\mu = \phi\sigma_3\phi^\dagger. \tag{1.92}$$

We immediately see that these two space-time vectors, which were known to be orthogonal and with opposite scalar squares, are now part of a (D_0, D_1, D_2, D_3) list – attention, please, as this is a major change in the Dirac theory, first obtained by D. Hestenes [75] – formed by four space-time vectors:

$$\mathrm{D}_0 := \mathrm{J} \ ; \quad \mathrm{D}_1 := \phi\sigma_1\phi^\dagger \ ; \quad \mathrm{D}_2 := \phi\sigma_2\phi^\dagger \ ; \quad \mathrm{D}_3 := \mathrm{K}. \tag{1.93}$$

21. It is the same for any quantity of the kind $\psi\overline{\psi}$, which are thus not general, despite the opinion of many physicists [68]. The preferred argument, based on the dimension of the linear space of Dirac matrices, with dimension 16 on the complex field, has no reason to apply here since the tensor densities are real ones. This 16 is actually a difference between triangular numbers ($36-10-10$, where $36 = 9\times8/2$, $10 = 5\times4/2$). The numerous studies based only on these 16 densities [10][93][104] miss an essential point. Moreover, the tensor densities without derivatives are not the only important densities in the Dirac theory. Some others with derivatives are used, while others were still misunderstood [51]. Worse yet, the list of the tensor densities which exist from the electron wave is infinite [16]. Hence we cannot know everything about these tensor densities.

The components of D_1 and D_2 (since not gauge-invariant) cannot be linear combinations of the 16 densities known through the old formalism of 4×4 complex matrices. We must then consider this formalism as seriously incomplete, weak, misleading. On the contrary, and more helpful than just the simplification of calculations, the shift to Cl_3 allows us to discover new quantities which will prove very useful in the next chapters.

For any similitude R defined by a dilator M, the four vectors D_μ transform similarly:

$$\mathrm{D}'_\mu = \phi'\sigma_\mu\phi'^\dagger = (M\phi)\sigma_\mu(M\phi)^\dagger = M\phi\sigma_\mu\phi^\dagger M^\dagger = M\mathrm{D}_\mu M^\dagger. \tag{1.94}$$

We recall that in relativistic physics the tensors are defined and classed by the way that they transform in a Lorentz transformation. We then have no reason to be concerned by D_0 and D_3, and no reason to deny the possibility of the existence of D_1 and D_2. The four D_μ vectors transform like the space-time vector x. We then say they are **contravariant**. They are also vectors of the same length. Moreover, they are orthogonal to each other and form a **mobile basis** of space-time with

$$\begin{aligned}
&2\mathrm{D}_\mu \cdot \mathrm{D}_\nu = \mathrm{D}_\mu\widehat{\mathrm{D}}_\nu + \mathrm{D}_\nu\widehat{\mathrm{D}}_\mu = \phi\sigma_\mu\phi^\dagger\widehat{\phi^\dagger}\widehat{\sigma}_\nu\overline{\phi} + \phi\sigma_\nu\phi^\dagger\widehat{\phi^\dagger}\widehat{\sigma}_\mu\overline{\phi}\\
&\quad = \phi\sigma_\mu\rho e^{-i\beta}\widehat{\sigma}_\nu\overline{\phi} + \phi\sigma_\nu\rho e^{-i\beta}\widehat{\sigma}_\mu\overline{\phi} = \rho e^{-i\beta}\phi(\sigma_\mu\widehat{\sigma}_\nu + \sigma_\nu\widehat{\sigma}_\mu)\overline{\phi} = \rho e^{-i\beta}\phi 2\delta_{\mu\nu}\overline{\phi}\\
&\quad = 2\delta_{\mu\nu}\rho e^{-i\beta}\phi\overline{\phi} = 2\delta_{\mu\nu}\rho e^{-i\beta}\rho e^{i\beta};\ \mathrm{D}_\mu \cdot \mathrm{D}_\nu = \delta_{\mu\nu}\rho^2.
\end{aligned} \tag{1.95}$$

Of course since here we use the space-time of special relativity with the choice of a + sign for time, we have

$$\delta_{00} = 1\ ;\quad \delta_{11} = \delta_{22} = \delta_{33} = -1\ ;\quad \delta_{\mu\nu} = 0\ ,\ \mu \neq \nu. \tag{1.96}$$

Among these ten relations (1.95), only three were known from the old formalism:

$$\mathrm{J}\cdot\mathrm{J} = \rho^2\ ;\quad \mathrm{K}\cdot\mathrm{K} = -\rho^2\ ;\quad \mathrm{J}\cdot\mathrm{K} = 0. \tag{1.97}$$

Now for the tensor densities $S^{\mu\nu}$, we let:

$$S_3 := S^{23}\sigma_1 + S^{31}\sigma_2 + S^{12}\sigma_3 + S^{10}i\sigma_1 + S^{20}i\sigma_2 + S^{30}i\sigma_3. \tag{1.98}$$

And we proved (see details in A.4.3) that

$$S_3 = \phi\sigma_3\overline{\phi}. \tag{1.99}$$

We can see immediately that S_3 is one of the four:

$$S_\mu := \phi\sigma_\mu\overline{\phi},\ \mu = 0, 1, 2, 3. \tag{1.100}$$

So now we have met S_0, which will be called a_1, since (see A.4.1):

$$a_1 := S_0 = \phi\sigma_0\overline{\phi} = \phi\overline{\phi} = \rho e^{i\beta} = \det(\phi). \tag{1.101}$$

With the four contravariant vectors D_μ which each have four components, together with S_0 which has two, and the three S_j, $j=1,2,3$ which each have six components, we count 36 tensor densities without a derivative. This is much more than the 16 known from the old Dirac theory, and is evident proof of the incomplete character of the old formalism. We may notice that this 36 is, like 16, a square, but this is a numerical coincidence because 36 is actually a triangular number: in Clifford algebras the triangular numbers $n(n-1)/2$ often appear. Since the right and left spinors forming the electron wave are the fundamental quantities (and we will see this by the study of weak interactions in the following chapter), the true counting is as follows: with each spinor, the right one R^1 and the left one $\widehat{L}^1$ in (1.68), we obtain $4\times 5/2 = 10 = 4+6$ densities. Four of them form a space-time vector, the $6 = 3\times 4/2$ others form a space-time bivector. With the right spinor, the vector D_R^1 and the bivector S_R^1 satisfy [29]:

$$\mathrm{D}_R^1 := R^1\widetilde{R}^1;\ S_R^1 := R^1\sigma_1\overline{R}^1. \tag{1.102}$$

With the left spinor of the electron, the vector D_L^1 and the bivector S_L^1 satisfy

$$\mathrm{D}_L^1 := L^1\widetilde{L}^1;\ S_L^1 := L^1\sigma_1\overline{L}^1. \tag{1.103}$$

In his theory of the magnetic monopole, G. Lochak was the first to notice the fundamental role of the left and right currents [84, 85, 86, 87, 88, 89, 90, 91]. These currents have a zero scalar square; they are on the light cone because:

$$\begin{aligned}
0 &= R^1\overline{R}^1 = \overline{R}^1R^1 = \widetilde{R}^1\widehat{R}^1 = \widehat{R}^1\widetilde{R}^1,\\
0 &= L^1\overline{L}^1 = \overline{L}^1L^1 = \widetilde{L}^1\widehat{L}^1 = \widehat{L}^1\widetilde{L}^1,\\
\mathrm{D}_R^1\cdot\mathrm{D}_R^1 &= \mathrm{D}_R^1\widehat{\mathrm{D}}_R^1 = R^1(\widetilde{R}^1\widehat{R}^1)\overline{R}^1 = 0,\\
\mathrm{D}_L^1\cdot\mathrm{D}_L^1 &= \mathrm{D}_L^1\widehat{\mathrm{D}}_L^1 = L^1(\widetilde{L}^1\widehat{L}^1)\overline{L}^1 = 0.
\end{aligned} \tag{1.104}$$

Hence in the wave with spin 1/2 some of the quantities always have properties of light, even at small velocity, and also for an electron at rest. For this reason the approximation (1.16) which suppresses the relativistic invariance is nonsense from the point of view of high-energy physics. Moreover a complex number, however small its modulus may be, may be written in trigonometric form with an argument which can be the phase of a wave. Only zero has no argument.

The J current and the K current are the sum and difference of the chiral (right and left) currents, and the bivectors S_1 and S_2 are also combinations of S_R^1 and S_L^1:

$$\mathrm{J} = \mathrm{D}_0 = \mathrm{D}_R^1 + \mathrm{D}_L^1;\ \mathrm{K} = \mathrm{D}_3 = \mathrm{D}_R^1 - \mathrm{D}_L^1, \tag{1.105}$$

$$S_1 + iS_2 = 2S_R^1;\ S_1 - iS_2 = 2S_L^1;\ S_R^1 = R^1\sigma_1\overline{R}^1;\ S_L^1 = L^1\sigma_1\overline{L}^1. \tag{1.106}$$

We derived in [50] the following relation, which will be generalized in the subsequent chapters:

$$\rho^2 = a_1 a_1^* = 2\mathrm{D}_R^1 \cdot \mathrm{D}_L^1. \tag{1.107}$$

Because we have

$$\begin{aligned}
\mathrm{J}\cdot\mathrm{J} &= (\mathrm{D}_R^1 + \mathrm{D}_L^1)\cdot(\mathrm{D}_R^1 + \mathrm{D}_L^1)\\
&= \mathrm{D}_R^1\cdot\mathrm{D}_R^1 + 2\mathrm{D}_R^1\cdot\mathrm{D}_L^1 + \mathrm{D}_L^1\cdot\mathrm{D}_L^1 = 2\mathrm{D}_R^1\cdot\mathrm{D}_L^1,\\
\widetilde{R}^1\widehat{L}^1 &= a_1^*\frac{1+\sigma_3}{2};\ \widetilde{L}^1\widehat{R}^1 = a_1^*\frac{1-\sigma_3}{2},
\end{aligned} \tag{1.108}$$

and also

$$\begin{aligned}
2\mathrm{D}_R^1\cdot\mathrm{D}_L^1 &= \mathrm{D}_R^1\widehat{\mathrm{D}}_L^1 + \mathrm{D}_L^1\widehat{\mathrm{D}}_R^1 = R^1\widetilde{R}^1\widehat{L}^1\overline{L}^1 + L^1\widetilde{L}^1\widehat{R}^1\overline{R}^1\\
&= R^1 a_1^*\frac{1+\sigma_3}{2}\overline{L}^1 + L^1 a_1^*\frac{1-\sigma_3}{2}\overline{R}^1\\
&= a_1^*(R^1\overline{L}^1 + L^1\overline{R}^1) = a_1^* a_1 = \rho^2.
\end{aligned} \tag{1.109}$$

Besides tensor densities coming from one of the two spinors, we have 16 densities that come from the two spinors, the right one and the left one. This $16 = 2^4$ was the (wrong) maximum number of tensor densities allowed by the old Dirac formalism with complex matrices. Here comes the well known $2^4 = 1+4+6+4+1$ of Pascal's triangle. The 1+1 of the extremities gives a_1, the 4+4 gives the vectors D_1 and D_2, and the 6 is the number of components of S_3:

$$\begin{aligned}
a_1 = S_0 = R^1\overline{L}^1 + L^1\overline{R}^1;\ S_3 = R^1\overline{L}^1 - L^1\overline{R}^1,\\
\mathrm{D}_1 + i\mathrm{D}_2 = 2R^1\sigma_1\widetilde{L}_1;\ \mathrm{D}_1 - i\mathrm{D}_2 = 2L^1\sigma_1\widetilde{R}_1.
\end{aligned} \tag{1.110}$$

The use of Cl_3 is absolutely necessary because only the construction of the tensor densities from Cl_3 may be generalized, and we will do this in the next chapter. [22] We will see the importance of the $(\mathrm{D}_0,\ \mathrm{D}_1,\ \mathrm{D}_2,\ \mathrm{D}_3)$ orthogonal basis in Chapter 4. Many other densities may also be useful in the Dirac theory, for instance the densities with first derivatives used in the study of the energy–momentum.

1.3.3 Relativistic transformation of the densities

We already explained how the D_μ vectors transform: they are contravariant vectors ($D' = MDM^\dagger$). Moreover, these formulas of transformation are

22. Otherwise R. Boudet and D. Hestenes made too much use of the mobile orthonormal basis $(\mathrm{e}_0,\ \mathrm{e}_1,\ \mathrm{e}_2,\ \mathrm{e}_3)$ such as $\mathrm{D}_\mu = \rho\mathrm{e}_\mu$ [5] [75]. Consequently they have not seen the similarity between the four D_μ and the four S_μ.

automatically induced by the transformation of the ϕ wave. Next, in the similitude R induced by any fixed dilator M, the four S_μ quantities become

$$S'_\mu = \phi'\sigma_\mu\overline{\phi}' = M\phi\sigma_\mu\overline{M\phi} = M\phi\sigma_\mu\overline{\phi}\,\overline{M} = MS_\mu\overline{M}. \tag{1.111}$$

Since physics characterizes the tensors by their transformation under a change of reference frame, we see no necessity in distinguishing the different D_μ vectors which transform similarly whether or not they are invariant in the electric gauge. The same situation happens for the different S_μ quantities. For instance we get:

$$\begin{aligned}\rho' e^{i\beta'} &= S'_0 = MS_0\overline{M} = M\rho e^{i\beta}\overline{M} = \rho e^{i\beta}M\overline{M} = \rho e^{i\beta} r e^{i\theta},\\ \rho' &= r\rho\ ;\quad \beta' = \beta+\theta.\end{aligned} \tag{1.112}$$

If we restrict the similitudes to only be Lorentz transformations then ρ is invariant, not β. Even in the case where $\det(M) = 1$ we may have $\beta' = \beta+\pi$.

Numerous relations exist between the 36 tensor densities that are dependent on the only eight real parameters of the ϕ wave (see A.4.6). The number 36 is also the result of restrictions for any other possibility: products like R^1R^1 or $R^1\widehat{R}^1$ cannot transform relativistically, because the multiplication by $M^\dagger$ on the right side is not available. And several products cancel, for instance $R^1L^{1\dagger}$.

The equalities in (1.111) are **entirely new in the physics of tensors**, completely different from the relations for the transformation of antisymmetric tensors of rank 2, which should give: $S'^{\rho\sigma} = R^\rho_\mu R^\sigma_\nu S^{\mu\nu}$. Since R^ν_μ is quadratic in M and multiplies each space-time length by r, the presence of two R factors implies a multiplication by r^2, while (1.111) is quadratic in M and thus multiplies the lengths only by r. Moreover, the J and K currents are perfectly similar since they are simply the sum and difference of the left D^1_L and right D^1_R currents. But the old formalism of 4×4 complex matrices considers J as a space-time vector and K as a pseudovector in space-time, which is wholly inconsistent: D^1_R and D^1_L have of course the same geometric status, they are both contravariant vectors. J and K are also contravariant vectors, necessarily. This is a sufficient reason to only use the framework of the Clifford algebra Cl_3.

Hence the old and the new formalism – the former one with 4×4 complex matrices and ours using Cl_3 – are not at all equivalent. Only the Cl_3 algebra is complete and we will use this framework as the true one from now on.

1.4 The invariant form of the Dirac equation

The form invariance of the wave equation of the electron uses the differential operator $\nabla = \overline{M}\nabla'\widehat{M}$. Since $\phi' = M\phi$ implies $\overline{\phi}' = \overline{\phi}\,\overline{M}$, the factor $\overline{M}$ on the left side indicates a possible multiplication of the wave equation on the left side by $\overline{\phi}$. When and where $\rho \neq 0$ (and only in this case), $\phi = \phi(\mathrm{x})$ is invertible. Hence by multiplying on the left side by $\overline{\phi}$ the Dirac equation is equivalent to

$$0 = \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\overline{\phi}\phi. \tag{1.113}$$

We consider this equation as the true Dirac equation and we now explain why this form is "the invariant form of the Dirac equation": In a Lorentz similitude R defined by a dilator M in Cl_3 satisfying (1.34), we get (1.60) and (1.62), which imply that if we conserve the gauge invariance:

$$\overline{\phi}(\nabla\widehat{\phi})\sigma_{21} = \overline{\phi}(\overline{M}\nabla'\widehat{M}\widehat{\phi})\sigma_{21} = \overline{\phi}'(\nabla'\widehat{\phi}')\sigma_{21}, \tag{1.114}$$

$$\overline{\phi}qA\widehat{\phi} = \overline{\phi}\ \overline{M}q'A'\widehat{M}\widehat{\phi} = \overline{\phi}'q'A'\widehat{\phi}'. \tag{1.115}$$

The two left terms of (1.113) are then form-invariant, and the mass term is also invariant if we have

$$m\overline{\phi}\phi = m'\,\overline{\phi}'\phi' = m'\,\overline{\phi}\ \overline{M}M\phi = re^{i\theta}m'\,\overline{\phi}\phi, \tag{1.116}$$

which is equivalent to:

$$m = re^{i\theta}m'. \tag{1.117}$$

Of course if we restrict M to $\det(M) = 1$ we have $m = m'$. But we must take caution that for the extended invariance the proper mass is no longer invariant. This is an important change in our habits: it is well known that the quantum wave is necessarily invariant under the Poincaré group formed by all transformations of the complete Lorentz group, plus space-time translations. But the Dirac equation is form-invariant only under these translations and transformations of the restricted Lorentz group. The similitude group also does not change the orientation of space and the orientation of time. Thus this group does not contain the totality of the Poincaré group, and theorems based on properties of the Poincaré group cannot apply here. But of course the proper mass remains invariant so long as the transformation belongs to the Poincaré group ($r = 1$). Yet no longer is the proper mass invariant when the transformation does not belong to this group ($r \neq 1$). The mass term reads:

$$m\overline{\phi}\phi = m\Omega_1 + im\Omega_2, \tag{1.118}$$

and is hence the sum of a scalar and a pseudoscalar. The second term of the invariant Dirac equation (1.113) shows another peculiarity: it is a space-time vector that we have calculated in (B.32):

$$\overline{\phi}A\widehat{\phi} = A_\nu \mathrm{D}^\nu_\mu \sigma^\mu = V_\mu\sigma^\mu;\ V_\mu = A_\nu \mathrm{D}^\nu_\mu = A\cdot \mathrm{D}_\mu. \tag{1.119}$$

This gives also:

$$\overline{\phi}\sigma^\nu\widehat{\phi} = \mathrm{D}^\nu_\mu\sigma^\mu. \tag{1.120}$$

Only the first term of (1.113) is a general term in Cl_3, but we can also obtain some properties with

$$\overline{\phi}(\nabla\widehat{\phi}) = \frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) + (\overline{\phi}\nabla)\widehat{\phi}] + \frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) - (\overline{\phi}\nabla)\widehat{\phi}], \quad (1.121)$$

$$\frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) + (\overline{\phi}\nabla)\widehat{\phi}] = \frac{1}{2}\partial_\nu(\overline{\phi}\sigma^\nu\widehat{\phi}) = \frac{1}{2}\partial_\nu(\mathrm{D}_\mu^\nu\sigma^\mu) = \frac{1}{2}(\partial_\nu\mathrm{D}_\mu^\nu)\sigma^\mu$$
$$= \frac{1}{2}(\nabla\cdot\mathrm{D}_\mu)\sigma^\mu = v = v_\mu\sigma^\mu;\ 2v_\mu = \nabla\cdot D_\mu, \quad (1.122)$$

$$\frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) - (\overline{\phi}\nabla)\widehat{\phi}] = iw = iw_\mu\sigma^\mu, \quad (1.123)$$

where v and w are two space-time vectors since $v^\dagger = v$ and $(iw)^\dagger = -iw$. This gives

$$\overline{\phi}(\nabla\widehat{\phi})\sigma_{21} = (v+iw)\sigma_{21}$$
$$= (v_0 + v_1\sigma^1 + v_2\sigma^2 + v_3\sigma^3 + iw_0 + w_1 i\sigma^1 + w_2 i\sigma^2 + w_3 i\sigma^3)(i\sigma^3)$$
$$= -w_3 + v_2\sigma^1 - v_1\sigma^2 - w^0\sigma^3 + i(v^3 + w_2\sigma^1 - w_1\sigma^2 + v_0\sigma^3). \quad (1.124)$$

Hence the decomposition of the invariant form (1.113) of the Dirac equation in the basis $(1,\sigma^1,\sigma^2,\sigma^3,i,i\sigma^1,i\sigma^2,i\sigma^3)$ of Cl_3 yields this system of eight real equations:

$$0 = -w_3 + qA\cdot\mathrm{D}_0 + m\Omega_1, \quad (1.125)$$
$$0 = \frac{1}{2}\nabla\cdot\mathrm{D}_2 + qA\cdot\mathrm{D}_1, \quad (1.126)$$
$$0 = -\frac{1}{2}\nabla\cdot\mathrm{D}_1 + qA\cdot\mathrm{D}_2, \quad (1.127)$$
$$0 = w_0 + qA\cdot\mathrm{D}_3, \quad (1.128)$$
$$0 = \frac{1}{2}\nabla\cdot\mathrm{D}_3 + m\Omega_2, \quad (1.129)$$
$$0 = -w_2, \quad (1.130)$$
$$0 = w_1, \quad (1.131)$$
$$0 = \frac{1}{2}\nabla\cdot\mathrm{D}_0. \quad (1.132)$$

The first equation is exactly the equation of the Lagrangian density $\mathcal{L} = 0$ because of the following (the detailed calculation is in B.1.4):

$$\mathcal{L} = \frac{1}{2}\Big[\big(\overline{\psi}\gamma^\mu(-i\partial_\mu + qA_\mu)\psi\big) + \big(\overline{\psi}\gamma^\mu(-i\partial_\mu + qA_\mu)\psi\big)^\dagger\Big] + m\overline{\psi}\psi$$
$$= -w_3 + qA\cdot\mathrm{D}_0 + m\Omega_1. \quad (1.133)$$

We know that by varying the Lagrangian density $\mathcal{L}$ we obtain the Dirac wave equation. Moreover the fact that the Dirac equation is homogeneous [23]

23. Wave equations in quantum mechanics are linear. They are thus additive, which means if ϕ_1 and ϕ_2 are two solutions of the wave equation then $\phi_1 + \phi_2$ is also a solution.

implies that $\mathcal{L} = 0$ when the wave equation is satisfied. Here we have exactly the reciprocal situation; the equation $\mathcal{L} = 0$ is one of the eight real equations equivalent to the wave equation, and **the Lagrangian formalism is an automatic consequence of the wave equation**.

Any law of movement, in classical mechanics and in electromagnetism, may be obtained from a Lagrangian formalism. We now know that this results from the Lagrangian form and the universality of quantum mechanics. But where does it come from that quantum mechanics has a Lagrangian form? Here we see that this is completely automatic because the Lagrangian density is the scalar part of the wave equation, and because this Lagrangian density yields anew the whole wave equation. We will detail in Chapter 2 how the single equation of the scalar part gives once again the seven other real equations, not from a physical principle above the differential laws but simply as a consequence of the algebraic structure due to the geometry of space-time. Moreover the four real equations containing the symmetric part v of $\overline{\phi}(\nabla\widehat{\phi})$ are with the D_μ of (1.93) (see A.4.2):

$$0 = \nabla \cdot \mathrm{D}_0, \tag{1.134}$$

$$0 = \nabla \cdot \mathrm{D}_3 + 2m\Omega_2, \tag{1.135}$$

$$0 = \nabla \cdot \mathrm{D}_1 - 2qA \cdot \mathrm{D}_2, \tag{1.136}$$

$$0 = \nabla \cdot \mathrm{D}_2 + 2qA \cdot \mathrm{D}_1. \tag{1.137}$$

The equation (1.134) which is known as the law of conservation of the probability current, is now exactly one of the eight real equations equivalent to the Dirac equation. Next (1.135) is known as Uhlenbeck-Laporte relation. The real equations (1.136) and (1.137) show that the space-time vectors D_1 and D_2 are not gauge-invariant; the gauge transformation operates a rotation in the plane of D_1 D_2.

1.4.1 Charge conjugation

Many years after the discovery of the electron, the positron was also discovered. The only difference between electron and positron is the charge sign: negative for the electron, positive for the positron. From the Dirac wave of the particle (1.2) (where the wave of the electron is denoted as ψ_e and the wave of the positron is denoted as ψ_p), quantum mechanics derives the wave equation of the antiparticle as follows. The complex conjugation is used on the Dirac equation:

$$0 = [\gamma^{\mu *}(\partial_\mu - iqA_\mu) - im]\psi_e^*. \tag{1.138}$$

And they are homogeneous, which means that if ϕ is a solution of the wave equation and if z is any fixed number, then $z\phi$ is also a solution of the wave equation. Beware! The word "homogeneous" here has its usual meaning in mathematics and has nothing to do with the consideration of dimensions in physics. Additivity and homogeneity together make up the linearity of the wave equation.

Since (1.4) gives $\gamma_2\gamma^{\mu*} = -\gamma^\mu\gamma_2$, $\mu = 0, 1, 2, 3$, by multiplying (1.138) by $i\gamma_2$ on the left side, we get

$$0 = -[\gamma^\mu(\partial_\mu - iqA_\mu) + im]i\gamma_2\psi_e^*. \tag{1.139}$$

Then up to an arbitrary phase, quantum mechanics supposes [24] :

$$\psi_p = i\gamma_2\psi_e^*, \tag{1.140}$$

which gives

$$0 = [\gamma^\mu(\partial_\mu - iqA_\mu) + im]\psi_p. \tag{1.141}$$

This equation is exactly the same as the equation of the electron up to the change of the sign of the electric charge. We automatically obtain the equality between the mass of the particle and the mass of the antiparticle. Using the decomposition of ψ with left and right waves, and assigning e indices for the electron and p indices for the positron, the link (1.140) between the electron wave and the positron wave reads:

$$\begin{pmatrix}\xi_{1p}\\ \xi_{2p}\\ \eta_{1p}\\ \eta_{2p}\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\xi_{1e}^*\\ \xi_{2e}^*\\ \eta_{1e}^*\\ \eta_{2e}^*\end{pmatrix}. \tag{1.142}$$

This gives

$$\xi_{1p} = \eta_{2e}^*, \quad \xi_{2p} = -\eta_{1e}^*; \quad \eta_{1p} = -\xi_{2e}^*; \quad \eta_{2p} = \xi_{1e}^*. \tag{1.143}$$

Now the same calculation in space algebra, and always with e indices for the electron and p indices for the positron, uses:

$$\widehat{\phi}_e = \sqrt{2}\begin{pmatrix}\eta_{1e} & -\xi_{2e}^*\\ \eta_{2e} & \xi_{1e}^*\end{pmatrix}; \quad \widehat{\phi}_p = \sqrt{2}\begin{pmatrix}\eta_{1p} & -\xi_{2p}^*\\ \eta_{2p} & \xi_{1p}^*\end{pmatrix}. \tag{1.144}$$

Then (1.140), which is equivalent to (1.143), is also equivalent to

$$\widehat{\phi}_p = \widehat{\phi}_e\sigma_1; \quad \phi_p = -\phi_e\sigma_1. \tag{1.145}$$

Once again we must recall that the charge conjugation described here is completely equivalent to the charge conjugation described by all textbooks of quantum physics. Only the style of writing is changed.

24. Quantum mechanics uses γ_2 because it is the only Dirac matrix with imaginary terms while the three other γ_μ matrices are real, given (1.4). Moreover the relation (1.140) is, by (1.7), independent from the choice of the γ_μ matrices.

1.5 Improved invariant equation

We now come to our main departure from the Dirac theory which underlies all the relativistic components of the Standard Model. This change is also the main difference from Hestenes' work. He always used the linear Dirac equation, and only changed the framework and the presentation of this equation. **Here we change the wave equation itself**. It can be seen from equations in (1.89) that the invariant form of the Dirac equation (1.113) is given by

$$0 = \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\rho\cos(\beta), \tag{1.146}$$

The improvement that we introduced in [12] was the elimination of $\cos(\beta)$ and now we add the replacement of the scalar mass term m by a matrix term $\mathbf{m}$ [43] [44]:

$$0 = \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + \mathbf{m}\rho;\ \mathbf{m} = \begin{pmatrix} \mathbf{l} & 0 \\ 0 & \mathbf{r} \end{pmatrix} \tag{1.147}$$

where $\mathbf{l}$ is the left mass term and $\mathbf{r}$ is the right mass term. The existence of two possibly different masses will be corroborated by their consequences. This improved equation is equivalent to the system of real equations:

$$0 = -w_3 + qA \cdot \mathrm{D}_0 + m_a\rho;\ m_a := \frac{\mathbf{l}+\mathbf{r}}{2} \tag{1.148}$$

$$0 = \frac{1}{2}\nabla \cdot \mathrm{D}_2 + qA \cdot \mathrm{D}_1, \tag{1.149}$$

$$0 = -\frac{1}{2}\nabla \cdot \mathrm{D}_1 + qA \cdot \mathrm{D}_2, \tag{1.150}$$

$$0 = w_0 + qA \cdot \mathrm{D}_3 + d\rho;\ d := \frac{\mathbf{l}-\mathbf{r}}{2} \tag{1.151}$$

$$0 = \frac{1}{2}\nabla \cdot \mathrm{D}_3, \tag{1.152}$$

$$0 = -w_2, \tag{1.153}$$

$$0 = w_1, \tag{1.154}$$

$$0 = \frac{1}{2}\nabla \cdot \mathrm{D}_0. \tag{1.155}$$

This is actually a simplification since three of the eight equations equivalent to the Dirac equation have been simplified: In the first line the invariant $m\Omega_1 = m\rho\cos(\beta)$ is simply replaced by $m_a\rho$ (where m_a is the arithmetic mean). The fourth equation (1.151) now has a $d\rho$ term and becomes very similar to the first equation. This new and apparently slight modification to the Dirac theory does not change five of the eight equations. Nevertheless this slight simplification improves many things: the last change, in (1.152), means the existence of a second conservative current, the $\mathrm{K} = \mathrm{D}_3$ current.

Since J and K currents are now both conservative, their sum and difference are also conservative [25]: the D_R^1 and D_L^1 chiral currents are conservative. This will be generalized for the electroweak domain. Moreover the eight equations come two by two. This is a mere consequence of the left–right structure of the wave.

For a comparison between our new improved equation and the Dirac equation it is enough to multiply (1.147) by $\overline{\phi}^{-1}$ on the left side, which gives

$$0 = \nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + \rho\overline{\phi}^{-1}\mathbf{m}. \tag{1.156}$$

We implicitly supposed, when multiplying on the left side by $\overline{\phi}^{-1}$, that ϕ was invertible, which means $\rho \neq 0$. Then a unique element M_ϕ exists in $SL(2,\mathbb{C})$ satisfying:

$$\phi = \sqrt{\rho}e^{i\beta/2}M_\phi;\ \overline{\phi} = \sqrt{\rho}e^{i\beta/2}\overline{M}_\phi = \sqrt{\rho}e^{i\beta/2}M_\phi^{-1}. \tag{1.157}$$

Actually M_ϕ is an element of $SL(2,\mathbb{C})$, since we have

$$\begin{aligned}\det\phi = \rho e^{i\beta} = \phi\overline{\phi} = \sqrt{\rho}e^{i\beta/2}M_\phi\sqrt{\rho}e^{i\beta/2}\overline{M}_\phi = \rho e^{i\beta}M_\phi\overline{M}_\phi,\\ 1 = M_\phi\overline{M}_\phi = \det(M_\phi);\ \overline{M}_\phi = M_\phi^{-1}.\end{aligned} \tag{1.158}$$

The existence of this M_ϕ element of $SL(2,\mathbb{C})$ was obtained by G. Lochak as early as 1956 [92] and was obtained independently by Hestenes ten years later [73]. This M_ϕ was also the starting point of the work of R. Boudet [5][6]. None of these physicists saw the difference between the field of dilators ϕ (similar to M in 1.1.2) and the field of the induced similitudes (like R). We also get

$$\overline{\phi}^{-1} = \frac{e^{-i\beta/2}}{\sqrt{\rho}}M_\phi;\ \rho\overline{\phi}^{-1} = \sqrt{\rho}e^{-i\beta/2}M_\phi = e^{-i\beta}\phi. \tag{1.159}$$

Then when we compare with the former Dirac equation, the improved equation appears not with a term less, but with an additional term $e^{-i\beta}$:

$$0 = \nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + e^{-i\beta}\phi\mathbf{m}. \tag{1.160}$$

The usual Dirac equation (1.2) is thus the linear approximation of our improved equation (1.160) when the Yvon-Takabayasi β angle is null or negligible and when the difference d between left and right mass terms is null.

25. The conservation of the left and right currents was obtained as early as 1983 by Lochak in his theory of the leptonic magnetic monopole [84]–[91], the theory from which comes our mass term in a particular case, where the Dirac equation is the linear approximation of our equation. Our equation is nevertheless another, distinct wave equation, because we conserved the electric gauge term of the equation of the electron. This gauge term is different from the gauge term of Lochak's monopole. In his theory of the monopole, the invariance of the electric gauge is only global (weaker) and it is the chiral gauge that is local (stronger). On the contrary, for our improved equation the electric gauge is local and the chiral gauge is only global. Since Noether's theorem requires only global invariance, our equation, like Lochak's, has the same conserved currents.

Henceforth in the succeeding chapters, only the improved equation will be generalized. The properties of this improved equation are often simpler than those of the usual Dirac equation, and are closer to physical reality. This is what we explain now.

1.5.1 Uncrossed form of the wave equation

We incorporate in (1.160) the left and right waves defined in (1.68):

$$\begin{aligned}0 &= \nabla(\widehat{L}^1+\widehat{R}^1)(-i\sigma_3)+qA(\widehat{L}^1+\widehat{R}^1)+e^{-i\beta}(R^1+L^1)\mathbf{m} \\ &= (-i\nabla\widehat{L}^1+qA\widehat{L}^1+\mathbf{l}e^{-i\beta}R^1)+(i\nabla\widehat{R}^1+qA\widehat{R}^1+\mathbf{r}e^{-i\beta}L^1).\end{aligned} \tag{1.161}$$

In this last line, the left bracketed quantity is a matrix with two zeros in its second column, while the second bracketed quantity is a matrix with two zeros in its first column. This equation is hence equivalent to the system:

$$\begin{aligned}0 &= -i\nabla\widehat{L}^1+qA\widehat{L}^1+\mathbf{l}e^{-i\beta}R^1, \\ 0 &= i\nabla\widehat{R}^1+qA\widehat{R}^1+\mathbf{r}e^{-i\beta}L^1.\end{aligned} \tag{1.162}$$

And we have:

$$\begin{aligned}\mathrm{J} &= \phi\widetilde{\phi} = R^1\widetilde{R}^1+L^1\widetilde{L}^1, \\ \rho e^{-i\beta} &= \widehat{\phi}\widetilde{\phi} = \widehat{L}^1\widetilde{R}^1+\widehat{R}^1\widetilde{L}^1.\end{aligned} \tag{1.163}$$

We now consider the vector v [26] such that:

$$\mathrm{v} := \frac{1}{\rho}\mathrm{J} = \frac{1}{\rho}(R^1\widetilde{R}^1+L^1\widetilde{L}^1). \tag{1.164}$$

We then get:

$$\begin{aligned}\mathrm{v}\widehat{L}^1 &= \frac{1}{\rho}(R^1\widetilde{R}^1+L^1\widetilde{L}^1)\widehat{L}^1 = \frac{1}{\rho}R^1\widetilde{R}^1\widehat{L}^1+\frac{1}{\rho}L^1\widetilde{L}^1\widehat{L}^1 = \frac{1}{\rho}R^1\widetilde{R}^1\widehat{L}^1, \\ \widetilde{R}^1\widehat{L}^1 &= 2\begin{pmatrix}\xi_1^* & \xi_2^* \\ 0 & 0\end{pmatrix}\begin{pmatrix}\eta_1 & 0 \\ \eta_2 & 0\end{pmatrix} = \begin{pmatrix}a_1^* & 0 \\ 0 & 0\end{pmatrix};\ a_1 = \rho e^{i\beta}, \\ R^1\widetilde{R}^1\widehat{L}^1 &= R^1 a_1^*\frac{1+\sigma_3}{2} = a_1^* R^1, \\ \mathbf{l}e^{-i\beta}R^1 &= \frac{\mathbf{l}}{\rho}a_1^* R^1 = \frac{\mathbf{l}}{\rho}R^1\widetilde{R}^1\widehat{L}^1 = \mathbf{l}\mathrm{v}\widehat{L}^1.\end{aligned} \tag{1.165}$$

Similarly, we have:

$$\begin{aligned}\mathrm{v}\widehat{R}^1 &= \frac{1}{\rho}(R^1\widetilde{R}^1+L^1\widetilde{L}^1)\widehat{R}^1 = \frac{1}{\rho}L^1\widetilde{L}^1\widehat{R}^1 \\ &= \frac{1}{\rho}L^1\overline{\widetilde{R}^1\widehat{L}^1} = \frac{1}{\rho}L^1 a_1^*\frac{1-\sigma_3}{2} = \frac{a_1^*}{\rho}L^1 = e^{-i\beta}L^1.\end{aligned} \tag{1.166}$$

26. The notation for a vector in space-time is in Roman typeset when using Cl_3 and in bold letters when using $Cl_{1,3}$.

Thus the system (1.162) is equivalent to the uncrossed system:

$$\begin{aligned} 0 &= -i(\nabla + iqA + i\mathbf{l}\mathrm{v})\widehat{L}^1, \\ 0 &= -i(\widehat{\nabla} + iq\widehat{A} + i\mathbf{r}\widehat{\mathrm{v}})R^1. \end{aligned} \tag{1.167}$$

In this last equation, we used the conjugation $P : M \mapsto \widehat{M}$. It is to be noted that this system is not completely uncrossed because v is dependent on both the left and right parts of the wave. If we write this term using the Yvon-Takabayasi β angle and using the right spinor ξ^1 and the left spinor η^1, then we get an equivalent form:

$$\begin{aligned} 0 &= -i(\nabla + iqA)\eta^1 + \mathbf{l}e^{-i\beta}\xi^1, \\ 0 &= -i(\widehat{\nabla} + iq\widehat{A})\xi^1 + \mathbf{r}e^{i\beta}\eta^1. \end{aligned} \tag{1.168}$$

We can use either form since they are equivalent.

1.5.2 Gauge invariance

Since the differential term and the gauge term do not change when we shift from the usual Dirac equation to the improved equation, and since the mass term is gauge-invariant, the improved equation is also invariant under the electric gauge. This gauge is expressed in the Cl_3 algebra as:

$$\phi \mapsto \phi' = \phi e^{ia\sigma_3} \ ; \quad A \mapsto A' = A - \frac{1}{q}\nabla a. \tag{1.169}$$

As with the usual Dirac equation, the conservative current linked to the electric gauge invariance by Noether's theorem is the $\mathrm{J} = \mathrm{D}_0$ current. The first difference introduced by our improved Dirac equation is the status of this conservation law, which is now one of the eight real equations equivalent to the wave equation in invariant form. The second difference is the existence, among these eight equations, of another conservation law (1.152) for the $\mathrm{K} = \mathrm{D}_3$ current. This current comes from Lochak's theory of the magnetic monopole [84]–[91], at the origin of our improved wave equation. This second conservation law is linked to the global gauge invariance (chiral gauge):

$$\phi \mapsto \phi' = e^{ia}\phi \ ; \quad \overline{\phi} \mapsto \overline{\phi}' = e^{ia}\overline{\phi} \ ; \quad \partial_\mu a = 0, \tag{1.170}$$

which gives

$$\begin{aligned} \rho e^{i\beta} = \phi\overline{\phi} \mapsto \rho' e^{i\beta'} &= \phi'\overline{\phi}' = e^{2ia}\phi\overline{\phi} = \rho e^{i(\beta+2a)}, \\ \rho \mapsto \rho' &= \rho \ ; \quad \beta \mapsto \beta' = \beta + 2a. \end{aligned} \tag{1.171}$$

We name this invariance the "chiral gauge" since the generator of the gauge group is the i which orients space [27]. We will encounter this chiral gauge

27. In Cl_3, for any orthonormal (u, v, w) basis, this basis is direct if and only if $uvw = i$, and is inverse if and only if $uvw = -i$ (see A.3.1).

again in the study of weak interactions. It is also the gauge of Lochak's magnetic monopole. Since the chiral gauge multiplies ϕ by e^{ia}, therefore $\widehat{\phi}$ is multiplied by e^{-ia}, the ξ spinor that is the left column of ϕ is multiplied by e^{ia}, and η which is the left column of $\widehat{\phi}$ is multiplied by e^{-ia} [86].

The improved equation has lost the linearity of the usual Dirac equation because ρ depends on ϕ, and because the determinant which defines ρ and β is not linear in ϕ. The sum $\phi_1 + \phi_2$ of two solutions of (1.160) is not necessarily a solution of (1.160). On the contrary, since the equation is homogeneous and invariant on the chiral gauge, if ϕ is a solution and if z is any complex number then $z\phi$ is also a solution of (1.160). This property, common to the Schrödinger, Klein-Gordon and Pauli equations, is not true for the Dirac equation with the i which is a 3-vector in Cl_3. As we will see with the Pauli principle, this completely obscures the nonlinearity of the invariant wave equation, and then induces a false necessity for linearity in relativistic quantum physics. Moreover, the calculation of interferences with Young's slits, for instance, are not made with a single relativistic electron (the slits should be much narrower). Thus we are in the case where the approximation by the Pauli equation used by Gondran [94] is perfectly legitimate.

Indeed the Dirac equation in space algebra contains both ϕ and $\widehat{\phi}$. And if we multiply by i we must not forget that $\widehat{i} = -i$. The isomorphism existing between Cl_3 and $M_2(\mathbb{C})$ is only an isomorphism of algebras on the real field, not on the complex field. The multiplication by i in Cl_3, a pseudoscalar term, does not correspond to the multiplication by i into $M_2(\mathbb{C})$. On the contrary, this multiplication by i corresponds in Cl_3 to the multiplication on the right side by $i\sigma_3$, a term which is a 2-vector (an oriented area), not a 3-vector (an oriented volume). Since Cl_3 is isomorphic to $Cl_{1,3}^+$ this $i\sigma_3$ becomes in space-time algebra the $\gamma_1\gamma_2$ 2-vector used by Hestenes [74].

This restricted isomorphism is also the reason for the discordance between the earlier form of the Dirac equation, using α_k and β matrices, and the Dirac equation in Cl_3. The Hamiltonian formalism that we get with these matrices acts on a wave equation that is not relativistic, and moreover that is indeed equivalent to the Dirac equation expressed in $M_4(\mathbb{C})$ (with the unique i of quantum mechanics), but it cannot be equivalent to the Dirac equation expressed in Cl_3 or $Cl_{1,3}$, where the unique i is replaced by a 2-vector. The unique i of the Dirac theory, commuting with everything, is proof of the fact that the theory is expressed not in $Cl_{1,3}$ or $Cl_{3,1}$ but in $M_4(\mathbb{C})$ which is isomorphic with the Clifford algebras $Cl_{2,3}$ and $Cl_{4,1}$, thus in a space-time with five dimensions. Hence it is the algebra of a space-time with a non-physical supplementary dimension.

1.5.3 Plane wave

We take again (with the same reservations) the calculation made in 1.3.3 for the usual Dirac equation. Our improved equation is now reduced, for

$A = 0$, to
$$\nabla\widehat{\phi} + e^{-i\beta}\phi\mathbf{m}\sigma_{12} = 0. \tag{1.172}$$
We consider the same plane wave with the φ phase satisfying
$$\phi = \phi_0 e^{\varphi\sigma_{21}} \ ; \quad \varphi = m_g \mathrm{v}_\mu \mathrm{x}^\mu \ ; \quad \mathrm{v} = \sigma^\mu \mathrm{v}_\mu, \tag{1.173}$$
where v is a fixed reduced velocity ($\mathrm{v}\widehat{\mathrm{v}} = 1$) and ϕ_0 is also a constant factor, and hence we get
$$\nabla\widehat{\phi} = \sigma^\mu \partial_\mu(\widehat{\phi}_0 e^{\varphi\sigma_{21}}) = -m_g \mathrm{v}\widehat{\phi}\sigma_{12}. \tag{1.174}$$
Then (1.160) is equivalent to
$$0 = (-m_g\mathrm{v}\widehat{\phi} + e^{-i\beta}\phi\mathbf{m})\sigma_{12}; \ \phi\mathbf{m} = m_g e^{i\beta}\mathrm{v}\widehat{\phi}, \tag{1.175}$$
$$\widehat{\mathrm{v}}\phi\mathbf{m} = m_g e^{i\beta}\widehat{\mathrm{v}}\mathrm{v}\widehat{\phi} = m_g e^{i\beta}\widehat{\phi}, \tag{1.176}$$
Conjugating we get
$$\mathrm{v}\widehat{\phi}\widehat{\mathbf{m}} = m_g e^{-i\beta}\phi, \tag{1.177}$$
which implies
$$\phi = e^{i\beta}\mathrm{v}\widehat{\phi}\frac{\widehat{\mathbf{m}}}{m_g}; \ \widehat{\phi} = e^{-i\beta}\widehat{\mathrm{v}}\phi\frac{\mathbf{m}}{m_g}. \tag{1.178}$$
We then have:
$$\phi = e^{i\beta}\mathrm{v}\Big(e^{-i\beta}\widehat{\mathrm{v}}\phi\frac{\mathbf{m}}{m_g}\Big)\frac{\widehat{\mathbf{m}}}{m_g} = \phi\frac{\mathbf{m}\widehat{\mathbf{m}}}{m_g^2}. \tag{1.179}$$
Therefore if ϕ_0 is invertible we must take:
$$m_g^2 = \mathbf{m}\widehat{\mathbf{m}} = (m + d\sigma_3)(m - d\sigma_3) = m^2 - d^2 = \mathbf{lr},$$
$$m_g = \sqrt{\mathbf{lr}}. \tag{1.180}$$
Thus the mass term m_g is the geometric mean of the left and right mass terms ($m_g < m_a$ if $\mathbf{l} \neq \mathbf{r}$). Multiplying (1.176) by $\phi^\dagger$ on the right side we obtain:
$$\widehat{\mathrm{v}}\phi\mathbf{m}\phi^\dagger = m_g e^{i\beta}\widehat{\phi}\phi^\dagger, \tag{1.181}$$
$$\widehat{\mathrm{v}}(\mathbf{l}\mathrm{D}_R^1 + \mathbf{r}\mathrm{D}_L^1) = m_g e^{i\beta}\rho e^{-i\beta}, \tag{1.182}$$
$$\widehat{\mathrm{J}}(\mathbf{l}\mathrm{D}_R^1 + \mathbf{r}\mathrm{D}_L^1) = m_g\rho^2 = \sqrt{\mathbf{lr}}\widehat{\mathrm{J}}\mathrm{J}. \tag{1.183}$$
Dividing by $\widehat{\mathrm{J}}$ this implies:
$$\mathbf{l}\mathrm{D}_R^1 + \mathbf{r}\mathrm{D}_L^1 = \sqrt{\mathbf{lr}}(\mathrm{D}_R^1 + \mathrm{D}_L^1), \tag{1.184}$$
$$\mathbf{l}\mathrm{D}_R^{10} + \mathbf{r}\mathrm{D}_L^{10} = \sqrt{\mathbf{lr}}(\mathrm{D}_R^{10} + \mathrm{D}_L^{10}) = \sqrt{\mathbf{lr}}\rho\mathrm{v}^0. \tag{1.185}$$
Since $\mathrm{D}_R^{10} = |\xi_1^1|^2 + |\xi_2^1|^2 > 0$, $\mathrm{D}_L^{10} = |\eta_1^1|^2 + |\eta_2^1|^2 > 0$, $\sqrt{\mathbf{lr}} > 0$ and $\rho > 0$ we obtain:
$$\mathbf{l} > 0; \ \mathbf{r} > 0; \ \mathrm{v}^0 > 0. \tag{1.186}$$

We thus have, as in the case where $\mathbf{l} = \mathbf{r}$:

$$1 = \mathrm{v}\widehat{\mathrm{v}} = (\mathrm{v}^0 - \vec{\mathrm{v}})(\mathrm{v}^0 + \vec{\mathrm{v}}) = (\mathrm{v}^0)^2 - \vec{\mathrm{v}}^2, \tag{1.187}$$

$$\mathrm{v}^0 = \sqrt{1 + \vec{\mathrm{v}}^2}. \tag{1.188}$$

Thus we solve here in the simplest manner the old problem of unphysical negative energy: the plane wave of the electron may only have positive energy and positive proper mass (we will see later the question of charge conjugation). **The improved wave equation is thus much better than the linear Dirac equation**: the non-existence of negative energies, never observed in particle physics, does not need second quantization to find an explanation.

1.5.4 Extended invariance

We start from the invariant form (1.147). The similitude R induced by the dilator M with ratio $r = |\det(M)|$ satisfies

$$\begin{aligned} &\mathrm{x}' = R(\mathrm{x}) = M\mathrm{x}M^\dagger \;,\;\; \det(M) = re^{i\theta} \;,\;\; \phi' = M\phi, \\ &\nabla = \overline{M}\nabla'\widehat{M} \;;\;\; qA = \overline{M}q'A'\widehat{M}. \end{aligned} \tag{1.189}$$

We also have:

$$\begin{aligned} &\rho' e^{i\beta'} = \det(\phi') = \det(M\phi) = \det(M)\det(\phi) = re^{i\theta}\rho e^{i\beta} = r\rho e^{i(\beta+\theta)}, \\ &\quad \rho' = r\rho;\; \beta' = \beta + \theta. \end{aligned} \tag{1.190}$$

And we obtain:

$$\begin{aligned} 0 &= \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + \mathbf{m}\rho = \overline{\phi}\;\overline{M}\nabla'\widehat{M}\widehat{\phi}\sigma_{21} + \overline{\phi}\;\overline{M}q'A'\widehat{M}\widehat{\phi} + \mathbf{m}\rho \\ &= \overline{\phi}'(\nabla'\widehat{\phi}')\sigma_{21} + \overline{\phi}'q'A'\widehat{\phi}' + \mathbf{m}\rho. \end{aligned} \tag{1.191}$$

The improved equation is form-invariant under Cl_3^*, which is the multiplicative group of the invertible elements of Cl_3, if and only if:

$$\mathbf{m}\rho = \mathbf{m}'\rho';\; \mathbf{m}\rho = \mathbf{m}'r\rho. \tag{1.192}$$

We then obtain the form invariance of the wave equation under $Cl_3^* = GL(2, \mathbb{C})$ if and only if:

$$\mathbf{m} = \mathbf{m}'r;\; \mathbf{l} = \mathbf{l}'r;\; \mathbf{r} = \mathbf{r}'r;\; m_a = m'_a r;\; m_g = m'_g r;\; d = d'r. \tag{1.193}$$

These equalities are simpler than the $m = m're^{i\theta}$ that the usual Dirac equation gives – and this is a powerful argument for our improved equation.

What is the meaning of these equalities for physics? If the true invariance group of the electromagnetic laws is not only the Lorentz group, not even its covering group, but the stronger Cl_3^* group, similar things must happen as when Galilean physics was replaced by relativistic physics, which put together mass and momentum, electric field and magnetic field. The proper mass m_0 and the density $\rho = ||\mathrm{J}||$ are both invariant under Lorentz transformations. Under the similitudes induced by any dilator M, we find that $\mathbf{m}$ and ρ are no longer separately invariant, and only their product $\mathbf{m}\rho$ remains invariant:

$$\begin{aligned}\mathbf{m}\rho &= \mathbf{m}' r\rho = \mathbf{m}'\rho', \\ d\rho &= d' r\rho = d'\rho'.\end{aligned} \tag{1.194}$$

Hence only the product of a reduced mass and a ratio of similitude is fully invariant. And the reduced mass $m = m_0 c/\hbar$ is proportional to the inverse of length in space-time, which means a frequency. Let us consider an analogy for better understanding: the fact that gravitational acceleration is proportional to the acceleration due to inertia results in a constant ratio between gravitational mass and inertial mass (and so this ratio may be put equal to one). This is the starting point of Einstein's gravitation. Because when the scale parameter r changes arbitrarily, the ratio between ρ and $1/m = \hbar/m_0 c$ is constant, and this needs the existence of a constant, which is the Planck constant. We may then say that **the existence of the Planck constant is a consequence of invariance under the Cl_3^* group**, which is a greater group than the local invariance group of either special or general relativity. We will see in the next chapter how the quantization of the action is linked to invariance under Cl_3^*, a greater group than the invariance group of special relativity. From this point of view we may also say this: the existence of the Planck constant has not been fully understood. The consideration of a greater invariance group will allow us to see things differently, and will later allow us in chapter 2 to understand **why the kinetic momentum is quantized with the value $\hbar/2$.**

1.5.5 Normalization of the wave

We start from the improved wave equation with the system in (1.168), and we use:

$$\mathrm{J} = \mathrm{D}_L^1 + \mathrm{D}_R^1 = \rho \mathrm{v};\ \mathrm{J}\widehat{\mathrm{J}} = \rho^2;\ \mathrm{D}_L^{1\mu} = \eta^{1\dagger}\sigma^\mu\eta^1;\ \mathrm{D}_R^{1\mu} = \xi^{1\dagger}\widehat{\sigma}^\mu\xi^1. \tag{1.195}$$

We may express the Lagrangian density of the improved wave equation as:

$$\begin{aligned}\mathcal{L} = \frac{m}{k\mathbf{l}}\mathcal{L}_L + \frac{m}{k\mathbf{r}}\mathcal{L}_R;\ \mathcal{L}_L &= \Re[\eta^{1\dagger}(-i\nabla + qA + \mathbf{l}\mathrm{v})\eta^1], \\ \mathcal{L}_R &= \Re[\xi^{1\dagger}(-i\widehat{\nabla} + q\widehat{A} + \mathbf{r}\widehat{\mathrm{v}})\xi^1],\end{aligned} \tag{1.196}$$

where k is a constant which is further explained, because we have:

$$\begin{aligned}&\frac{m}{\mathbf{l}}\eta^{1\dagger}\mathbf{l}\mathrm{v}\eta^1 + \frac{m}{\mathbf{r}}\xi^{1\dagger}\mathbf{r}\widehat{\mathrm{v}}\xi^1 = me^{-i\beta}\eta^{1\dagger}\xi^1 + me^{i\beta}\xi^{1\dagger}\eta^1 \\ &= \frac{m}{2}e^{-i\beta}\rho e^{i\beta} + \frac{m}{2}e^{i\beta}\rho e^{-i\beta} = m\rho.\end{aligned} \tag{1.197}$$

With the covariant derivatives

$$d^L_\mu := -i\partial_\mu + qA_\mu + \mathbf{l}\mathrm{v}_\mu : \; d^R_\mu := -i\partial_\mu + qA_\mu + \mathbf{r}\mathrm{v}_\mu, \tag{1.198}$$

we can express the Lagrangian density as:

$$\mathcal{L} = \Re\Big[-i\Big(\frac{m}{k\mathbf{l}}\eta^{1\dagger}\sigma^\mu d^L_\mu\eta^1 + \frac{m}{k\mathbf{r}}\xi^{1\dagger}\widehat{\sigma}^\mu d^R_\mu\xi^1\Big)\Big]. \tag{1.199}$$

The invariance of the Lagrangian density under the space-time translations, like in the linear Dirac theory, implies the existence of a conservative energy–momentum tensor density, Tétrode's T. Since the wave equation is homogeneous the Lagrangian density is null for any solution of the wave equation and Tétrode's tensor density is expressed as:

$$\begin{aligned} T^\mu_\nu &= \Re\Big[-i\Big(\frac{m}{k\mathbf{l}}\eta^{1\dagger}\sigma^\mu d^L_\nu\eta^1 + \frac{m}{k\mathbf{r}}\xi^{1\dagger}\widehat{\sigma}^\mu d^R_\nu\xi^1\Big)\Big] - \delta^\mu_\nu\mathcal{L} \\ &= \Re\Big[-i\Big(\frac{m}{k\mathbf{l}}\eta^{1\dagger}\sigma^\mu d^L_\nu\eta^1 + \frac{m}{k\mathbf{r}}\xi^{1\dagger}\widehat{\sigma}^\mu d^R_\nu\xi^1\Big)\Big]. \end{aligned} \tag{1.200}$$

For a wave with an energy E satisfying

$$-id^L_0\eta^1 = \frac{E}{\hbar c}\eta^1; \; -id^R_0\xi^1 = \frac{E}{\hbar c}\xi^1, \tag{1.201}$$

we get

$$\begin{aligned} T^0_0 &= \Re\Big[-i\Big(\frac{m}{k\mathbf{l}}\eta^{1\dagger}d_0\eta^1 + \frac{m}{k\mathbf{r}}\xi^{1\dagger}d_0\xi^1\Big)\Big] \\ &= \frac{E}{\hbar c}\Big(\frac{m}{k\mathbf{l}}\eta^{1\dagger}\eta^1 + \frac{m}{k\mathbf{r}}\xi^{1\dagger}\xi^1\Big) = E\frac{\mathbf{J}^0}{\hbar c}, \end{aligned} \tag{1.202}$$

$$\mathbf{J} := \frac{m}{k\mathbf{l}}\mathrm{D}^1_L + \frac{m}{k\mathbf{r}}\mathrm{D}^1_R.. \tag{1.203}$$

The condition for normalization of the wave function:

$$\iiint dv\frac{\mathbf{J}^0}{\hbar c} = 1, \tag{1.204}$$

is then equivalent to

$$E = \iiint dvT^0_0. \tag{1.205}$$

The left term of this sum is the total energy E of the electron, which de Broglie conceived of as a very small clock with frequency $E = h\nu$, while the right term is the sum of the local energy density of the electron. We will see that this local density is linked to inertia through the Lorentz force. Hence **it is not because we must have a probability that the wave must be normalized**. The physical wave is normalized, always, because the inertial mass-energy acted on by the exterior forces is equal to the absolute value of the gravitational mass-energy. So this energy has a determined value, not an arbitrary one. The normalization of the electron wave that is a law in quantum mechanics [28] is thus the equivalence between gravitational mass and inertial mass, the principle at the basis of general relativity. The existence of a probability density, for any electron wave and in any possible case, is not a principle on which any physical theory must be built: It is simply the necessary equality between inertial mass and gravitational mass. And this is the same whether for the usual Dirac equation, or for the improved equation which the usual Dirac equation linearly approximates [22].

Since T^{μ}_{ν} must have the dimension of an energy density, ML^2T^{-2}/L^3, and since $\mathbf{J}$ has the dimension $\hbar c/L^3 = M/T^2$, $k\mathbf{J}$ has dimension: $\dim(k)M/T^2$, which is the dimension $\dim(\phi)^2$. Then ϕ is without physical dimension if and only if k has dimension T^2/M. We therefore suppose from now on that k has this dimension T^2/M.

Normalization obviously applies to the solutions for the hydrogen atom that we study in Appendix C [29]. Moreover, since the $|\psi|^2$ of the Schrödinger wave is a particular case of the approximation of the Dirac wave by a part of its components, the need for normalizing the wave function of an electron – which is part of the principles of nonrelativistic quantum theory – follows as in the relativistic case from the equivalence principle. This is very important for the unification of all interactions, because until now the existence of probabilities in quantum mechanics was thought of as a metaphysical principle governing any present and future theory, while in fact this is only the consequence of the equality between gravitational and inertial mass. At the same time we understand better why Bohr was able to rebut all of Einstein's arguments against Born's probabilistic interpretation: The existence of a probability density comes from gravitation.

The probabilities that Einstein was thinking of derive from thermodynamics, in which case there is not only one particle, but myriads of particles moving in all directions. Furthermore we used the expression "probability density" and we carefully avoided the expression "probability of presence." The first expression makes sense because the theory of probabilities, like integral calculus, was developed from the same mathematics, measure theory. The second expression cannot make sense because any experimental verification of the probability of presence of the electron-particle, for instance a probability of 0.1 in a domain D of space, supposes that we can attain

28. This normalization is so important that it was included among the postulates imposed on any quantum wave. In fact normalization is allowed by the wave equations but is not deduced from them. It is the cause of great difficulties, like the collapse of the ψ or Schrödinger's dead–living cat. The issue of normalization also precipitated the setback of de Broglie's pilot–wave and afterward Bohm's.

29. We must recall that the density $\mathbf{J}^0$ is not equal to the relativistic invariant ρ, which is the norm of another vector: $\mathrm{J}^0 \neq \mathbf{J}^0$. It is the time component of a space-time vector. We also recall that T^0_0 is a component of a nonsymmetric tensor. It is well known that the integration of the spin 1/2 into relativistic gravitation is only possible with a nonzero torsion [89] (see also Chapter 4).

the convergence of statistical frequency at 0.1. And though it is possible to obtain the statistics from the myriads of photons moving on a single light wave, it is absolutely impossible to obtain statistics from the single electron that can occupy an electron wave. It is possible to obtain statistics from electrons only if we have a great number of them and each one necessarily has its own particular wave. We can say that the probability of the domain D is 0.1 if the sum over D of the probability density is 0.1, and this is all that may be said about the wave of an electron. For a system of electrons it is necessary to use the Pauli principle that we will study later. Now we can see why any scientific discussion about probabilities in quantum physics needs very careful phrasing. A general theoretical discussion about probabilities has little meaning: properties of electrons, each one being alone on its wave, are radically different from properties of photons which can move on the same wave. For instance the violation of Bell's inequalities was experimentally observed only for photons. For electrons this remains unproved. Entanglement needs at least two waves.

1.5.6 Charge conjugation

We again begin with $\phi_p = -\phi_e\sigma_1$, the link between the wave of the particle and the wave of the antiparticle in relativistic quantum mechanics. The improved wave equation (1.160) reads for the particle:

$$\nabla\widehat{\phi}_e\sigma_{21} + qA\widehat{\phi}_e + e^{-i\beta_e}\phi_e\mathbf{m} = 0. \tag{1.206}$$

We also have

$$\rho_e e^{i\beta_e} = \phi_e\overline{\phi}_e. \tag{1.207}$$

This then gives:

$$\rho_e e^{i\beta_e} = \phi_e\overline{\phi}_e = \phi_p(-\sigma_1)(\widehat{\phi}_p\sigma_1)^\dagger = -\phi_p\overline{\phi}_p = -\rho_p e^{i\beta_p}. \tag{1.208}$$

Therefore (1.206) takes the form:

$$\nabla\widehat{\phi}_p\sigma_1\sigma_{21} + qA\widehat{\phi}_p\sigma_1 + (-e^{-i\beta_p})(-\phi_p\sigma_1\mathbf{m}) = 0. \tag{1.209}$$

Multiplying on the right side by σ_1, this is equivalent to

$$\begin{aligned} 0 &= -\nabla\widehat{\phi}_p\sigma_{21} + qA\widehat{\phi}_p + e^{-i\beta_p}\phi_p\widehat{\mathbf{m}}, \\ 0 &= \nabla\widehat{\phi}_p\sigma_{21} - qA\widehat{\phi}_p - e^{-i\beta_p}\phi_p\widehat{\mathbf{m}}. \end{aligned} \tag{1.210}$$

Next, multiplying on the left side by $\overline{\phi}_p$, we get the invariant wave equation of the antiparticle:

$$\begin{aligned} 0 &= -\overline{\phi}_p\nabla\widehat{\phi}_p\sigma_{21} + q\overline{\phi}_pA\widehat{\phi}_p + \widehat{\mathbf{m}}\rho_p, \\ 0 &= \overline{\phi}_p\nabla\widehat{\phi}_p\sigma_{21} - q\overline{\phi}_pA\widehat{\phi}_p - \widehat{\mathbf{m}}\rho_p \end{aligned} \tag{1.211}$$

The first equation means that the lone differential term of the wave equation changes sign. This was the reason for Feynman [69] to interpret charge conjugation as parity–time (PT) symmetry. The second equation means that the charge term changes sign, and that the mass term is changed in a way that the arithmetic average $m = (\mathbf{r}+\mathbf{l})/2$ changes sign while the difference $\mathbf{l}-\mathbf{r}$ remains unchanged, so we can say that the role of left and right parts are exchanged. Next we are able to recast the previous equations as:

$$0 = \nabla\widehat{\phi}_p\sigma_{12} + qA\widehat{\phi}_p + e^{-i\beta_p}\phi_p\widehat{\mathbf{m}}. \tag{1.212}$$

Plane wave

The improved wave equation is reduced, if $A=0$, to:

$$0 = -\nabla\widehat{\phi}_p + e^{-i\beta_p}\phi_p\widehat{\mathbf{m}}\sigma_{12}. \tag{1.213}$$

We consider a solution such as:

$$\phi_p := \phi_0 e^{\varphi_p\sigma_{21}};\ \varphi_p := m_g \mathrm{v}_{p\mu}\mathrm{x}^\mu;\ \mathrm{v}_p := -\mathrm{v}. \tag{1.214}$$

We obtain the same results as in 1.5.3 :

$$m_g = \sqrt{\mathbf{lr}};\ \mathrm{v}^0 = \sqrt{1+\vec{\mathrm{v}}^2}. \tag{1.215}$$

$$\mathrm{v}_p^0 = -\sqrt{1+\vec{\mathrm{v}}_p^2} \tag{1.216}$$

Hence we again obtain plane wave solutions with a negative time coefficient, necessary for Fourier transformation and for very small wave packets, but with a positive mass-energy, in accordance with experiment. Created with the same energy, the electron and positron are moving with opposite velocity vectors, which is also in accordance with experiment.

Numeric equations

Multiplying on the left side by $\overline{\phi}_p$ we get the invariant wave equation:

$$0 = \overline{\phi}_p\nabla\widehat{\phi}_p\sigma_{12} + q\overline{\phi}_pA\widehat{\phi}_p + \widehat{\mathbf{m}}\rho_p;\ \widehat{\mathbf{m}} = \begin{pmatrix}\mathbf{r} & 0\\ 0 & \mathbf{l}\end{pmatrix}. \tag{1.217}$$

Nonrelativistic quantum mechanics, using a single i, could not truly understand charge conjugation, which simply changes the sign of the $\sigma_{21} = \sigma_2\sigma_1$ term into $\sigma_{12} = \sigma_1\sigma_2$ and the sign of the difference d. The first change of sign is thus only a change of direction in the series of σ_k, which is also a change of orientation, the left wave becoming right and conversely, and changing also the sign of d. Instead of the system of the eight equations of the particle, we now have the same system, albeit one where the components of all ∂_μ and d change sign:

$$0 = w_3 + qA\cdot \mathrm{D}_0 + m_a\rho, \tag{1.218}$$

$$0 = -\frac{1}{2}\nabla \cdot \mathrm{D}_2 + qA \cdot \mathrm{D}_1, \tag{1.219}$$
$$0 = +\frac{1}{2}\nabla \cdot \mathrm{D}_1 + qA \cdot \mathrm{D}_2, \tag{1.220}$$
$$0 = -w_0 + qA \cdot \mathrm{D}_3 - d\rho, \tag{1.221}$$
$$0 = -\frac{1}{2}\nabla \cdot \mathrm{D}_3, \tag{1.222}$$
$$0 = w_2, \tag{1.223}$$
$$0 = -w_1, \tag{1.224}$$
$$0 = -\frac{1}{2}\nabla \cdot \mathrm{D}_0. \tag{1.225}$$

Charge conjugation changes the sign of the charge and the sign of chiral masses, because we cannot change the arrow of time nor the orientation of space [59]. Actually, only the differential terms of the wave equations and d change sign. The electric gauge invariance is now obtained as:

$$\phi_p \mapsto \phi'_p = \phi_p e^{ia\sigma_3}, \tag{1.226}$$
$$A \mapsto A' = A - \frac{1}{q}(-\nabla a) = A - \frac{1}{-q}\nabla a.$$

Thus the positron seems to have a charge opposite to that of the electron. But in fact it is not q but $\partial_\mu a$ that changes sign. And so only ∂_μ, v_μ, w_μ and d change sign. With the covariant derivatives

$$\overline{d}^R_\mu = i\partial_\mu + qA_\mu + \mathbf{r}\mathrm{v}_\mu;\ \overline{d}^L_\mu = i\partial_\mu + qA_\mu + \mathbf{l}\mathrm{v}_\mu, \tag{1.227}$$

we can express the Lagrangian density as:

$$\overline{\mathcal{L}} = \Re\Big(\frac{m}{k\mathbf{r}}\eta_p^{1\dagger}\widehat{\sigma}^\mu \overline{d}^R_\mu \eta^1_p + \frac{m}{k\mathbf{r}}\xi_p^{1\dagger}\overline{d}^L_\mu \sigma^\mu \xi^1_p\Big). \tag{1.228}$$

The normalization of the wave is thus, always given a stationary state, equivalent to:

$$\iiint dvT^0_0 = -E. \tag{1.229}$$

The positive mass-energy of the positron is exactly the opposite of the negative energy-coefficient of the stationary wave. **The improved wave equation thus resolves the problem of the energy sign in a way that is much easier to understand than second quantization**: we have the negative coefficients $-|E|$ necessary to obtain the Fourier transformation, and the true energy density is the T^0_0 component of the energy–momentum that remains positive. Since the wave equation of the antiparticle is obtained from that of the particle simply by changing ∂_μ into $-\partial_\mu$ and d into $-d$, which also result from the PT transformation, the CPT theorem of quantum field theory is trivial. Therefore, charge conjugation is the purely quantum and purely relativistic phenomenon of a wave which in a sense sees space-time upside down. This was the point of view of Feynman [69]. And since the usual Dirac equation is the linear approximation of our improved equation, we derive the Dirac equation of the positron from the improved equation of the positron by changing the mass term: we must account for the fact that $\beta_p = \beta_e + \pi$ and that $\Omega_{1e} = -\Omega_{1p}$. The linear approximation of the improved wave equation of the positron satisfies $m = \mathbf{l} = \mathbf{r}$, which implies:

$$
\begin{aligned}
0 &= -\overline{\phi}_p \nabla \widehat{\phi}_p \sigma_{21} + q\overline{\phi}_p A \widehat{\phi}_p - m\Omega_{1p}, && (1.230)\\
0 &= -\nabla \widehat{\phi}_p \sigma_{21} + qA\widehat{\phi}_p - m\phi_p,\\
0 &= \nabla \widehat{\phi}_p \sigma_{21} + (-q)A\widehat{\phi}_p + m\phi_p. && (1.231)
\end{aligned}
$$

This is precisely the Dirac equation of the positron, with the charge appearing with a changed sign. We have, for the sign of E and of T_0^0, the same results as with the improved equation: E is negative while T_0^0 is positive.

1.5.7 The hydrogen atom

Early quantum mechanics obtained the quantization of the energy levels by solving the Schrödinger equation in the case of the hydrogen atom, an electron “revolving” around a proton. Obtaining the quantization was a brilliant result. But the other results were not as good. For instance the energy levels were not very precise. And the total number of quantum states for the principal quantum number $\mathbf{n}$ was thought to be $\mathbf{n}^2$ when the actual number should be $2\mathbf{n}^2$ states.

The detailed calculation using our improved equation is presented in Appendix C. This calculation is quite different from the one used in the early years of quantum mechanics. At that time the theory of proper values and proper vectors in Hermitian spaces was developed mainly for application to the angular momentum operators. Bohr understood the Mendeleev periodic table by counting all possible values of the angular momentum of atomic electron particles. This gave Bohr the expected energy levels $k/\mathbf{n}^2$, but not the expected number of states. Next Sommerfeld looked into relativistic dynamics of particles to obtain more states, as well as the fine structure of atomic spectral lines. Quantum mechanics replaced this counting with the calculation of solutions of the electron wave equation, which are proper vectors of operators with the same algebraic properties as classical angular momentum. Using the Schrödinger wave equation, only Bohr’s model was reproduced. The relativistic Klein-Gordon equation was able to obtain the second quantum number introduced by Sommerfeld, but not with the correct values, because integer numbers (0, 1, 2, ...), which are the only possible values with angular momentum operators, must be replaced by half-integer values (1/2, 3/2, 5/2, ...) to account for spectroscopic lines. This profound divergence between theory and experiments on light led to the hypothesis of the spin of the electron, and thus the Pauli wave equation and finally the Dirac equation. Only this relativistic equation was able to obtain the true quantum numbers and energy levels in the case of the hydrogen atom. The set of adequate solutions was obtained by C.G. Darwin immediately after the discovery of the wave equation by Dirac, using kinetic momentum operators that issued from the Pauli equation.

As for us, we use a completely different method obtained by H. Krüger [81], a marvelous classical one from the mathematical point of view, separating the variables in spherical coordinates (r, θ, φ). A new calculation is made in Appendix C to account for left and right mass terms. The principal quantum number $\mathbf{n}$ remains a sum: $\mathbf{n} = |\kappa| + n$ (see [99]); κ is introduced in the separation between the variables $\mathrm{x}^0 = ct$ and φ, from the variables θ and radial r. The study of functions of θ relocates the normalization of the wave to conditions governing κ and n: κ must be a nonzero integer number (positive or negative) and n is the degree of the polynomial functions included in the series expansion of radial functions. It is the necessity of normalizing the solution of the wave equation that allows us the existence of a probability density, which gives the κ and n integers, and thus the $\mathbf{n}$ number. The last quantum number, here called λ, is obtained from the sole condition that the wave must be a well-defined function in space-time, with unique value in Cl_3. There is absolutely no need for operators of angular momentum, even if the quantity j defined as $j = |\kappa| - 1/2$ and the quantum number λ have exactly the same algebraic properties as the l number and the n number obtained via spherical harmonics of nonrelativistic quantum mechanics: it is possible to construct operators such that these $j(j+1)$ and λ are proper values. But each λ is a half-integer, never an integer. Let us see how, without an integer angular momentum l, the various states are obtained.[30] Consider for instance the case $\mathbf{n} = 5$, where if $\kappa > 0$ we may have:

1. $n = 4$ and $\kappa = 1$, thus $j = 1/2$, and 2 states: $\lambda = -1/2$ and $\lambda = 1/2$.
2. $n = 3$ and $\kappa = 2$, thus $j = 3/2$, and 4 states: $\lambda = -3/2$, $-1/2$, $1/2$ and $3/2$.
3. $n = 2$ and $\kappa = 3$, thus $j = 5/2$, and 6 states: $\lambda = -5/2$, $-3/2$, $-1/2$, $1/2$, $3/2$ and $5/2$.
4. $n = 1$ and $\kappa = 4$, thus $j = 7/2$, and 8 states: $\lambda = -7/2$, $-5/2$, $-3/2$, $-1/2$, $1/2$, $3/2$, $5/2$ and $7/2$.
5. $n = 0$ and $\kappa = 5$, thus $j = 9/2$, and 10 states: $\lambda = -9/2$, $-7/2$, $-5/2$, $-3/2$, $-1/2$, $1/2$, $3/2$, $5/2$, $7/2$ and $9/2$.

30. The angular momentum l of nonrelativistic quantum mechanics cannot be a constant of the movement in relativistic quantum mechanics. The classification of atoms should hence never use the integer number l, which is nevertheless always used in course books on chemistry for the sake of pretty “orbitals” that are far removed from true electron physics!

This gives $2+4+6+8+10=30=5\times 6$ (more generally $\mathbf{n}\times(\mathbf{n}+1)$) states, which are two by two orthogonal (as was already explained by L. de Broglie in 1934 [54]). Since κ may also be negative, we should get the same set of quantum numbers, but if $n=0$ the states with $\kappa<0$ are exactly the same as those with $\kappa>0$, and this gives:

6. $n=4$ and $\kappa=-1$, thus $j=1/2$, and 2 states: $\lambda=-1/2$ and $\lambda=1/2$.

7. $n=3$ and $\kappa=-2$, thus $j=3/2$, and 4 states: $\lambda=-3/2$, $-1/2$, $1/2$ and $3/2$.

8. $n=2$ and $\kappa=-3$, thus $j=5/2$, and 6 states: $\lambda=-5/2$, $-3/2$, $-1/2$, $1/2$, $3/2$ and $5/2$.

9. $n=1$ and $\kappa=-4$, thus $j=7/2$, and 8 states: $\lambda=-7/2$, $-5/2$, $-3/2$, $-1/2$, $1/2$, $3/2$, $5/2$ and $7/2$.

This gives $2+4+6+8=20=5\times 4$ (more generally $\mathbf{n}\times(\mathbf{n}-1)$) states, two by two orthogonal and also orthogonal to each state with $\kappa>0$. In the end, we actually obtain $\mathbf{n}\times(\mathbf{n}+1)+\mathbf{n}\times(\mathbf{n}-1)=2\mathbf{n}^2$ states that are two by two orthogonal. Therefore Pauli's explanation of this number as the $\mathbf{n}^2$ from the Schrödinger equation multiplied by the "two values" of the spin, is nowadays only a tale for children. Unhappily this tale is still a popular one, because the theory of group representations, for systems of electrons, uses $SU(2)$ instead of $SL(2,\mathbb{C})$, because this group has no finite-dimensional unitary representation.

Our study of the solutions of the improved equation indicates that a family of solutions exists. These solutions are labeled by the same quantum numbers given by the usual Dirac equation. They are very near the solutions of the linear equation such that the Yvon-Takabayasi angle is everywhere defined and small. And if ϕ_1 and ϕ_2 are two solutions of this family, then $z_1\phi_1$ and $z_2\phi_2$ are also solutions of the improved equation because it is homogeneous and invariant under the chiral gauge. But the sum $z_1\phi_1+z_2\phi_2$ has no reason to be a solution since the determinant which has the Yvon-Takabayasi angle as argument is quadratic on the wave, and the angle has no reason to be small and negligible. The solutions labeled by the quantum numbers j, κ, λ and n thus give the only stationary states of the hydrogen atom. This explains why an electron in a hydrogen atom is usually in one of the labeled states, not in a linear combination of such states. **This is a well-known experimental fact that only the improved equation explains in a simple way**. This improved wave is thus closer to physical reality than its linear approximation which is the Dirac equation. Our improved equation is, as far as we know, the only nonlinear wave equation such that quantized levels of energy may exist with exactly the true energy levels.

It is well known that the Dirac equation was perfect for the electron, yet nevertheless two properties were not obtained: the anomalous magnetic moment and the Lamb shift. We examine now how this effect may be integrated to the solutions of our wave equation.

Lamb effect

This effect is a shift between the energy levels predicted as equal by Sommerfeld's formula and the Dirac theory, which are observed not equal: for instance the shift between the $\mathbf{2}s1/2$ and $\mathbf{2}p1/2$ states. The biggest shift is between the $\mathbf{1}s1/2$ calculated by the Dirac theory and the same level calculated by quantum field theory [72]. And the improved equation is actually able to calculate exactly the solutions integrating this effect: for the $\mathbf{1}s1/2$ states the important shift $(8,2$ GHz$)$ between the level calculated by Sommerfeld's formula and the observed level is consistent with a value of ν very close to 1: $1+6.615\times10^{-11}$. For the $\mathbf{2}p1/2$ states, whose energy level is in accordance with Sommerfeld's formula with $n=1$, the value of ν is much nearer 1, the shift from 1 is only 10^{-14}. To account for the shift between that states and the $\mathbf{2}s1/2$, a little upper, the value of ν obtained in C.3 is: $1+9.29\times10^{-12}$. We may remark that the value of ν verges on 1 when the quantum number n increases. The shift between $\mathbf{l}$ and $\mathbf{r}$ is hence very small, that explains why it was unknown. Even if very small, this shift between ν and 1, in the case of the ground state where it is maximal, could contribute to explain the shift between the theoretical value of the anomalous magnetic momentum with the observed momentum of the electron.

1.5.8 The Pauli principle

The main success of the usual Dirac equation is the calculation of the electron states in atoms. This calculation alone does not give all that was predicted by the atomic spectroscopy and by chemistry. There another principle is used, the Pauli exclusion principle, which says: **two electrons cannot be in the same quantum state** characterized by a set of quantum numbers $\mathbf{n}$, κ, j, λ, n. And two distinct solutions of the Dirac equation for the hydrogen atom are not only normalized, they are moreover orthogonal for the Hermitian scalar product of quantum mechanics. This is also true for the Euclidean scalar product characteristic of real Clifford algebras (details of this orthonormalization are already in [54], in later books like [99], and in our own works [14]). We now simply recall how the scalar product and the associated norm are linked: the norm of ϕ or of ψ is defined by integrating the probability density over all space. This integration is essential for obtaining the quantization of the energy levels:

$$||\phi|| = ||\psi|| = \iiint dv \frac{1}{\hbar c}\mathrm{J}^0;\ \mathrm{J} = \mathrm{J}^\mu \sigma_\mu = \phi\phi^\dagger. \tag{1.232}$$

Any norm on a linear space is associated with a scalar product and, conversely, any scalar product defines a norm. For any A and B vectors of the considered linear space, and with the notation $A\cdot B$ for the scalar product, we have:

$$||A|| = \sqrt{A \cdot A};\ 4(A \cdot B) = ||A + B||^2 - ||A - B||^2. \tag{1.233}$$

This is true for the solutions ϕ^+ and ϕ^- of the Dirac equation corresponding to $\kappa > 0$ and $\kappa < 0$ respectively, for the same set of quantum numbers λ (magnetic quantum number), j (kinetic momentum of the electron) and n (degree of the radial polynomial), which are approximations of the solutions to the improved equation for the hydrogen atom, the functions

$$\phi_+ = \frac{1}{\sqrt{2}}(\phi^+ + \phi^-);\ \phi_- = \frac{1}{\sqrt{2}}(\phi^+ - \phi^-), \tag{1.234}$$

are exactly those calculated by Darwin in 1928 [11][14]. This property results from the resolution of the Dirac equation obtained from a set of Dirac matrices γ'_μ as in (1.8). We also recall that our set of γ_μ matrices is the most suitable set for weak interactions and high velocity, and also convenient to find a simple connection between the Pauli algebra and space-time algebra. We previously saw that for the single-column ψ matrix this is equivalent to considering sums and differences multiplied by $1/\sqrt{2}$. The solutions are normalizable and quantization results from imposing the normalization. Solutions indeed exist aside from these normalizable ones, which have all the required properties. Therefore we can see that quantization of the energy levels comes from the condition of normalizing the states, which means:

$$1 = ||\phi_+|| = ||\phi_-||. \tag{1.235}$$

Since the $M = \frac{1}{\sqrt{2}}(\gamma_0 + \gamma_5)$ matrix transforming the γ_μ into the γ'_μ is equal to its inverse, Darwin's solutions are also sums and differences of our solutions:

$$\phi^+ = \frac{1}{\sqrt{2}}(\phi_+ + \phi_-);\ \phi^- = \frac{1}{\sqrt{2}}(\phi_+ - \phi_-), \tag{1.236}$$

The direct and complete proof of the orthogonality of the solutions was detailed in de Broglie's first book on the Dirac equation [54]. This proof is not at all trivial; it studies all cases and exploits the orthogonality of the Legendre polynomials and Laguerre polynomials. The ϕ_+ and ϕ_- states calculated by Darwin have a norm of one and are 2×2 orthogonal. The states that are the linear approximations of the solutions of our improved equation are the ϕ^+ and ϕ^- also with norm one and 2×2 orthogonal. The orthogonality of these states is a consequence of the normalization of Darwin's states, and similarly the orthogonality of Darwin's states is a consequence of the normalization of our solutions. Hence we get:

$$4\phi^+ \cdot \phi^- = ||\phi^+ + \phi^-||^2 - ||\phi^+ - \phi^-||^2 = 2||\phi_+||^2 - 2||\phi_-||^2 = 0. \tag{1.237}$$

Since the improved equation cannot have other stationary states than those listed by $\mathbf{n}$, κ, j, λ, n, only those states that are listed are possible. Since these states are 2×2 orthogonal, the Pauli principle may be so reformulated: **two electrons must occupy two orthogonal states** for the scalar product associated with the norm (1.232). Let now ϕ_1 and ϕ_2 be solutions of the Dirac equation corresponding to two electrons, with orthogonal and normalized states with a respective mass-energy E_1 and E_2. We denote as J_1 and J_2 the respective current associated with each solution, and T_1 and T_2 the energy–momentum tensor densities associated with each electron. We then have:

$$\iiint dv \frac{\mathrm{J}_1^0}{\hbar c} = 1; \iiint dv T_{1}{}_0^0 = -E_1; \ \phi_1 \cdot \phi_1 = 1, \tag{1.238}$$

$$\iiint dv \frac{\mathrm{J}_2^0}{\hbar c} = 1; \iiint dv T_{2}{}_0^0 = -E_2; \ \phi_2 \cdot \phi_2 = 1, \tag{1.239}$$

$$\phi_1 \cdot \phi_2 = 0. \tag{1.240}$$

From the bilinearity of the scalar product we get:

$$(\phi_1 + \phi_2) \cdot (\phi_1 + \phi_2) = \phi_1 \cdot \phi_1 + \phi_2 \cdot \phi_2 = 1 + 1 = 2, \tag{1.241}$$

$$\iiint dv \frac{\mathrm{J}^0}{\hbar c} = 2 = \iiint dv \frac{\mathrm{J}_1^0}{\hbar c} + \iiint dv \frac{\mathrm{J}_2^0}{\hbar c}, \tag{1.242}$$

$$\mathrm{J}^0 = \mathrm{J}_1^0 + \mathrm{J}_2^0; \ T_0^0 = T_{10}^0 + T_{20}^0, \tag{1.243}$$

$$\iiint dv (T_1 + T_2)_0^0 = \iiint dv T_{10}^0 + \iiint dv T_{20}^0 = -(E_1 + E_2). \tag{1.244}$$

The Pauli principle then implies that the mass-energy of an electron in a system is additive, that it is the sum of the energies of the electrons of this system. This additivity is essential for any theory integrating the different interactions to gravitation: this allows us to link gravity to the total mass of a star, both for Newtonian gravitation and for GR. From $E = \hbar\nu$, this also allows us to get a quantum wave with a frequency ν for any system of electrons. When only the electric phase of the wave is important (low velocity, negligible spin effects), it is then possible to reduce the quantum wave to a function with value in $\mathbb{C}$. This addition of the energies is translated into the addition of the phases, and hence into the product of the corresponding complex numbers. This allows us to express the Pauli principle as the anti-symmetrization of the product. But this anti-symmetrization is not general; it is conditioned by the reduction of the quantum wave to its nonrelativistic approximation.

If we agree that $q\mathrm{J}$ is space-time vector whose time component $q\mathrm{J}^0$ is the charge density and $q\vec{\mathrm{J}}$ is the electric current density, the global condition (1.242) is sufficient to obtain the electrostatic laws: the electric charge of a system of n electrons is ne.

We will now see how the anti-symmetrization that is nowadays the usual presentation of the Pauli principle in quantum mechanics is linked to the condition (1.243). Since J is the sum of the right current and the left current while the wave is also the sum of a right and a left wave, corresponding to two columns of the same matrix, the additivity of the J current is equivalent to the additivity of the D_R^1 and D_L^1 currents in any point of space-time [31]. Now we detail the case of the right current D_R^1: we denote the right currents as D_R^{11} and D_R^{12}, linked respectively to each electron state. We have:

$$R^{11} = \sqrt{2}\begin{pmatrix}\xi_1^{11} & 0\\ \xi_2^{11} & 0\end{pmatrix};\ R^{12} = \sqrt{2}\begin{pmatrix}\xi_1^{12} & 0\\ \xi_2^{12} & 0\end{pmatrix}. \tag{1.245}$$

We shall have $R^1 = R^{11} + R^{12}$ and $\mathrm{D}_R^1 = \mathrm{D}_R^{11} + \mathrm{D}_R^{12}$ if and only if:

$$(R^{11} + R^{12})(R^{11} + R^{12})^\dagger = R^{11}R^{11\dagger} + R^{12}R^{12\dagger}, \tag{1.246}$$

$$R^{11}R^{12\dagger} = -R^{12}R^{11\dagger} \tag{1.247}$$

$$2\begin{pmatrix}\xi_1^{11} & 0\\ \xi_2^{11} & 0\end{pmatrix}\begin{pmatrix}\xi_1^{12*} & \xi_2^{12*}\\ 0 & 0\end{pmatrix} = -2\begin{pmatrix}\xi_1^{12} & 0\\ \xi_2^{12} & 0\end{pmatrix}\begin{pmatrix}\xi_1^{11*} & \xi_2^{11*}\\ 0 & 0\end{pmatrix}, \tag{1.248}$$

which means if and only if the products are antisymmetric:

$$\begin{aligned}\xi_1^{11}\xi_1^{12*} &= -\xi_1^{12}\xi_1^{11*};\ \xi_2^{11}\xi_1^{12*} = -\xi_2^{12}\xi_1^{11*}\\ \xi_1^{11}\xi_2^{12*} &= -\xi_1^{12}\xi_2^{11*};\ \xi_2^{11}\xi_2^{12*} = -\xi_2^{12}\xi_2^{11*}.\end{aligned} \tag{1.249}$$

This anti-symmetrization, which induces the anti-symmetrization of non-relativistic quantum mechanics, was here obtained as a consequence of the additivity of the charge densities and of electric current densities. We may now reverse the presentation: from the anti-symmetrization of the products (1.249), which is the Pauli principle of anti-symmetrization, we obtain the additivity of the right and left currents D_R et D_L, and thus also both the additivity of the J current which gives the additivity of the electric current, and the additivity of the **J** current which gives the additivity of the mass-energy.

About probabilities

The wave equation of the electron is a wave equation for a single object: the Pauli principle absolutely forbids placing more than one electron on an electron wave. When physics needs the calculation of the probability of emission or absorption (a function of time), this includes not only one electron but a vast number of different electrons, each with an electron wave. *A priori* there is no connection between these probabilities and the probability density $\mathbf{J}^0$ (a function of space). If such a connection exists, this must be proven from the theoretical point of view and validated experimentally by statistics.

1.5.9 Iterative form of the improved equation

We calculate $\widehat{\mathrm{J}}\phi$:

$$\widehat{\mathrm{J}}\phi = \widehat{\phi\widehat{\phi}}\phi = \widehat{\phi}\rho e^{i\beta}. \tag{1.250}$$

Conjugating we then get

$$\mathrm{J}\widehat{\phi} = \rho e^{-i\beta}\phi;\ e^{-i\beta}\phi = \frac{\mathrm{J}}{\rho}\widehat{\phi}. \tag{1.251}$$

With (1.107) and (1.108), we define the unit vector:

$$\mathrm{v} = \mathrm{v}^\mu\sigma_\mu = \frac{\mathrm{J}}{\rho};\ \mathrm{v}\cdot\mathrm{v} = \mathrm{v}\widehat{\mathrm{v}} = \frac{\mathrm{J}\widehat{\mathrm{J}}}{\rho^2} = \frac{\rho^2}{\rho^2} = 1;\quad \widehat{\mathrm{v}} = \mathrm{v}^{-1}. \tag{1.252}$$

Similarly we obtain the mass term:

$$\mathbf{m}\widehat{\mathbf{m}} = \begin{pmatrix} \mathbf{l} & 0 \\ 0 & \mathbf{r} \end{pmatrix}\begin{pmatrix} \mathbf{r} & 0 \\ 0 & \mathbf{l} \end{pmatrix} = \begin{pmatrix} \mathbf{lr} & 0 \\ 0 & \mathbf{rl} \end{pmatrix} = m_g^2 \tag{1.253}$$

This allows us to express the improved wave equation as follows[50][46]:

$$\widehat{\phi} = -\widehat{\mathrm{v}}(\nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi})\frac{\widehat{\mathbf{m}}}{m_g^2}. \tag{1.254}$$

Conjugating, this gives the recursive functional equation:

$$\phi = f(\phi); f(\phi) = -\mathrm{v}(\widehat{\nabla}\phi\sigma_{21} + q\widehat{A}\phi)\frac{\mathbf{m}}{m_g^2}. \tag{1.255}$$

We then get by iterating:

$$\phi = f(f(\phi));\ \phi = f(f(f(\phi))), \dots \tag{1.256}$$

To get only multiplication by the left side, we use the uncrossed form (1.167):

$$\begin{aligned} &i\nabla\eta^1 = qA\eta^1 + \mathbf{l}\mathrm{v}\eta^1;\ i\nabla\eta^1 = p_L\eta^1;\ p_L := qA + \mathbf{l}\mathrm{v}, \\ &i\widehat{\nabla}\xi^1 = q\widehat{A}\xi^1 + \mathbf{r}\widehat{\mathrm{v}}\xi^1;\ i\widehat{\nabla}\xi^1 = \widehat{p}_R\xi^1;\ \widehat{p}_R := q\widehat{A} + \mathbf{r}\widehat{\mathrm{v}}. \end{aligned} \tag{1.257}$$

This gives at the second order:

$$i\widehat{\nabla}\nabla\eta^1 = \widehat{\nabla}(p_L\eta^1);\ i\nabla\widehat{\nabla}\xi^1 = \nabla(\widehat{p}_R\xi^1). \tag{1.258}$$

Iterating the wave equation once, we obtain second derivatives, and we then use the d'Alembert operator:

$$\Box = \widehat{\nabla}\nabla = \nabla\widehat{\nabla} = \partial^2_{00} - \partial^2_{11} - \partial^2_{22} - \partial^2_{33}. \tag{1.259}$$

We indeed get:

$$\begin{aligned} &i\Box\eta^1 = (\widehat{\nabla}p_L)\eta^1 + 2p_L^\mu\partial_\mu\eta^1 - \widehat{p}_L\nabla\eta^1, \\ &i\Box\xi^1 = (\nabla(\widehat{p_R})\xi^1) + 2p_R^\mu\partial_\mu\xi^1 - p_R\widehat{\nabla}\xi^1. \end{aligned} \tag{1.260}$$

We let:

$$F := \nabla\widehat{A};\ F_l := \nabla\widehat{p}_L;\ F_r := \nabla\widehat{p}_R. \tag{1.261}$$

This gives

$$i(\Box - p_L\cdot p_L)\eta^1 = \widehat{F}_l\eta^1 + 2p_L^\mu\partial_\mu\eta^1, \tag{1.262}$$

$$i(\Box - p_R\cdot p_R)\xi^1 = F_r\xi^1 + 2p_R^\mu\partial_\mu\xi^1. \tag{1.263}$$

These equations are more similar on the left side, but this is only because the d'Alembert operator suppresses the difference between $\widehat{\nabla}\nabla$ and $\nabla\widehat{\nabla}$. Two remarks: these equations are not the Klein-Gordon equations, as in the linear Dirac equation case, which also gives the $A^\mu\partial_\mu$ term. (This fact is rather well concealed in many books where the second-order wave equation is given for an electron without interaction, and thus without real physical existence.) Next, the electromagnetic field is introduced as two chiral fields F_l and F_r. We have:

$$\nabla\widehat{\mathrm{v}} = \nabla(\frac{\widehat{\mathrm{J}}}{\rho}) = \nabla(\frac{\rho\widehat{\mathrm{J}}}{\rho^2}) = \nabla(\frac{\rho\widehat{\mathrm{J}}}{\mathrm{J}\widehat{\mathrm{J}}}) = \nabla(\rho\mathrm{J}^{-1}) := G. \tag{1.264}$$

These fields satisfy

$$F_l = qF + \mathbf{l}G;\ F_r = qF + \mathbf{r}G. \tag{1.265}$$

1.6 Three generations

Both the wave equation of the electron (1.160) and the wave equation of the positron (1.212) contain a σ_{jk} at the right side of the gradient operator. This term is not unique but is also physically significant. This factor is one of six similar $\sigma_{jk} = \sigma_j\sigma_k,\ j\neq k$ terms. First the charge conjugation divides this number by two, because the wave equation with σ_{kj} is used in the wave equation of the antiparticle. This is the reason for the existence of three wave equations; the interpretation that is now made is the existence of three "generations" of fermions: besides electrons there also exist muons and taus. These waves have very similar properties; in particular they see the electromagnetic field in exactly the same way:

$$0 = \nabla\widehat{\phi}\sigma_{32} + qA\widehat{\phi} + e^{-i\beta}\phi\mathbf{m};\ 0 = \nabla\widehat{\phi}_p\sigma_{23} + qA\widehat{\phi}_p + e^{-i\beta_p}\phi_p\widehat{\mathbf{m}}. \tag{1.266}$$

$$0 = \nabla\widehat{\phi}\sigma_{13} + qA\widehat{\phi} + e^{-i\beta}\phi\mathbf{m};\ 0 = \nabla\widehat{\phi}_p\sigma_{31} + qA\widehat{\phi}_p + e^{-i\beta_p}\phi_p\widehat{\mathbf{m}}. \tag{1.267}$$

The discovery of the muon is now well in the past, yet quantum field theory still has no simple explanation for either the existence of the muon or of the tau. No further explanation accounts for why there are only three generations. Besides, the muon is not an electron: their properties are different. Meanwhile, we can stress the following point: the third direction with the

Dirac wave is completely privileged – when the spin of the electron is measured, this is always made in the third direction. When the solutions for the hydrogen atom are calculated, these solutions do not have the spherical symmetry of the potential; they only have an axial symmetry around the third direction. Of course with equation (1.266) the first direction becomes the privileged one. And with equation (1.267) the second direction becomes the privileged one. But this is not the only thing that differs when the generation changes. For instance as we used σ_1 to obtain the charge conjugation, then it will be necessary to replace this factor by σ_2 for the second generation and by σ_3 for the third generation.

Why has such a simple and evident explanation not come to light much earlier? The main cause seems to come from nonrelativistic quantum mechanics: the ground level of electrons in atoms is viewed as a function with spherical symmetry. The other states giving the orbitals used by chemistry, replacing the orbits of Bohr's atom, are obtained with spherical harmonics. Globally, they have the symmetry of the sphere. Thus the image that anyone finds in physics textbooks for the structure of an atom is depicted as electrons revolving in any direction. These images are never presented like a solar system with planets revolving in the same plane. We now make three remarks: firstly, in the case of the hydrogen atom, the lowest energy states for an electron following the Dirac equation do not have spherical symmetry. For each state a flow of the probability current exists in the plane orthogonal to the third direction. Secondly, the explanation of the anomalous Zeeman effect, from the Dirac equation, allows us to exactly obtain the shifting of the energy levels and the intensity of light rays, but only if the magnetic field is exactly oriented in the third direction. Finally, we must look at how the isotropy of space is re-established in the Dirac theory: it is said that the privileged direction is corrected by the fact that this third direction can be set in any direction of space by an appropriate rotation. And if we consider a system of electrons such that for each electron the third axis is set in a direction Ox and an M matrix such as $M = e^{ia\mathbf{u}}$, where a is any real number and $\mathbf{u}$ is a unit space vector, then $M\sigma_3 M^\dagger$ will be found in any direction. But we will also get $\phi' = M\phi$ for all electron states: they will then have the same axis of symmetry and the whole system will continue to have cylindrical symmetry, not spherical symmetry.

We now consider a wave following (1.266), with an axis of symmetry in the first direction. After the rotation defined by M the wave also becomes $\phi' = M\phi$, and this axis of symmetry is still orthogonal to the axis of symmetry of the electron system. The physical consequence for a muon entering into an electron system is that this muon is not governed by Pauli's exclusion principle. The muon does not see the wave of the other electrons – only their charges. The muon wave is a function of space-time which belongs to a linear subspace different from the subspace of the electron wave functions, with an unconditional additivity of the currents (see the Pauli principle in 1.5.8).

1.7 The numeric–dimension (dinum)

We define the numeric–dimension of a quantity as the power of the ratio of similitude r in the R transformation of the quantity generated by the dilator M in Cl_3. We abbreviate this as "dinum."

1. Since ϕ becomes $\phi' = M\phi$ and $M = \sqrt{r}e^{i\theta}P$, where P belongs to $SL(2,\mathbb{C})$, the dinum of ϕ is 1/2. The dinum of $\widehat{\phi}$, of $\widetilde{\phi}$ and of $\overline{\phi}$ is also 1/2. The dinum of ϕ^{-1} is $-1/2$.

2. Next, a contravariant vector such as x or $\mathrm{J} = \mathrm{D}_0 = \phi\phi^\dagger$, which transforms into $\mathrm{J}' = M\mathrm{J}M^\dagger$ has a dinum +1.

3. A covariant vector like $\nabla = \overline{M}\nabla'\widehat{M}$ has a dinum -1.

4. Since we have $m = m'r$ and $\rho' = r\rho$, we may say that ρ has the dinum 1 and m has the dinum -1.

5. Since space and time vary in the same way, any velocity has a zero dinum.

6. Since an acceleration is the derivative of a velocity its dinum is -1.

7. The electromagnetic potential A, in the second-order equation, is linked to the J current in a scalar product. This vector must be, like J, a contravariant vector. We may also use the fact that the electromagnetic potential is linked to its sources, which are the particles having an electric charge (or other charges: magnetic, strong and so on). Hence A must have a dinum +1 and must satisfy

$$A' = MAM^\dagger. \tag{1.268}$$

8. So that the gauge invariance may be compatible with relativistic invariance, qA must transform like a covariant vector while A is contravariant. We thus have:

$$\begin{aligned} qA &= \overline{M}q'A'\widehat{M} = \overline{M}q'MAM^\dagger\widehat{M} = q're^{i\theta}Are^{-i\theta} = r^2q'A, \\ q &= q'r^2;\ q' = qr^{-2}. \end{aligned} \tag{1.269}$$

The dinum of q is thus -2. We may remark that m and q do not have the same dinum. This is an important difference between mass and charge which have the same status in relativistic invariance but not in the extended invariance under Cl_3^*.

9. Next, we have:

$$q = \frac{e}{\hbar c}\ ;\ qe = \frac{e^2}{\hbar c} = \alpha = q'e'\ ;\quad qe = q'r^2e = q'e'. \tag{1.270}$$

$$e' = r^2e. \tag{1.271}$$

An electric (or magnetic) charge thus has a dinum +2, which is also the dinum of a surface.

10. We thus have:

$$\alpha = \frac{e^2}{\hbar c} = \frac{e'^2}{\hbar' c} = \frac{r^4 e^2}{\hbar' c} \; ; \quad e^2 \hbar' c = \hbar c r^4 e^2. \tag{1.272}$$
$$\hbar' = r^4 \hbar. \tag{1.273}$$

The "Planck constant" is thus a variable when the ratio of similitude is not reduced to 1 and the dinum of the action is 4 – this is the dinum of a space-time volume (this is convenient for relativistic thermodynamics). We also remark that it is not consistent to give to J^0 a status of probability density, but it is consistent for $\mathrm{J}^0/\hbar c$ which has the expected dinum -3.

11. For a proper mass m_0 we have

$$\frac{m_0 c}{\hbar} = m = rm' = r\frac{m'_0 c}{\hbar'} = r\frac{m'_0 c}{r^4 \hbar} = \frac{m'_0 c}{r^3 \hbar}. \tag{1.274}$$

And this gives

$$m'_0 = r^3 m_0. \tag{1.275}$$

A proper mass thus has a dinum +3; this is the dinum of a volume. Both a charge or a proper mass are no longer invariant, and this requires a change in our habits. Among the bad habits needing a quick change is taking $\hbar = 1$. This is nonsense if $\hbar$ is variable, as shown in (1.273). All these variations do not contradict relativistic invariance in the restricted sense, which is the particular case where $r = 1$: the concept of dinum is not pertinent in this case[31].

12. Pressure, with dimension $\mathrm{ML}^{-1}\mathrm{T}^{-2}$ thus has a null dinum.

13. We now consider the classical part $F = \nabla \widehat{A}$ of the electromagnetic field. We have:

$$F = \overline{M} \nabla' \widehat{M} \widehat{A},$$
$$MFM^{-1} = M\overline{M} \nabla' \widehat{M} \widehat{A} M^{-1} = re^{i\theta} \nabla' \widehat{M} \widehat{A} M^{-1} \tag{1.276}$$
$$= \nabla' \widehat{M} \widehat{A} r e^{i\theta} M^{-1} = \nabla' \widehat{M} \widehat{A} \overline{M} M M^{-1} = \nabla' \widehat{A}' = F'. \tag{1.277}$$

31. The variation of r changing m_0 and $\hbar$ has no consequence on measurements of mass, which are always measurements of the ratio between two masses. When physics passes from classical mechanics into relativistic mechanics, where masses are no longer invariant, there is no need to change the mass unit: any measurement of mass is obtained at zero velocity in the laboratory. It is the same here, because any proper mass and any action varies with the same ratio (r^3 for a proper mass, r^4 for an action), in the laboratory at the time when the measurement is made. The variation of $\hbar$, which remains relativistically invariant, is thus perfectly compatible with the replacement of the standard mass by a standard action, more accurate and more stable than the previous International Prototype Kilogram (IPK).

Since M brings $\sqrt{r}$ and since M^{-1} brings a $1/\sqrt{r}$ factor, the electromagnetic field has a dinum 0 (and this is necessarily the same for the other gauge fields). All these results are consistent with the laws of mechanics and of electromagnetism: mass, energy and momentum have the same dinum +3. A mechanical or electromagnetic force has a dinum 2: this is consistent with the force exerted on a charge since the dinum of a charge is 2 and the dinum of a field is 0.

The fact that the dinum of gauge fields is null, and the fact that they transform following the $F' = MFM^{-1}$ law is very important, as this implies that an F_1F_2 product of two such fields again satisfies the same rule:

$$F_1'F_2' = MF_1M^{-1}MF_2M^{-1} = MF_1F_2M^{-1}. \tag{1.278}$$

This is why products of photon fields may be added together and may follow Bose-Einstein statistics (actually found in the thesis of L. de Broglie [53]). In the kind of fields acting as operators on the wave, this also allows us the definition of creation and annihilation operators.

1.8 Invariant space-time

When we presented this double space-time in the book of the same name [22], we implicitly worked with ρ on an equal footing with r. This is natural since $\rho' = r\rho$. More generally there is no difference of structure between a dilator M defining the similitude R, and $\phi(\mathrm{x})$, which are both complex 2×2 matrices, which means two elements of the Cl_3 algebra. More precisely ϕ is a function of space-time with value in Cl_3. Therefore ϕ, like M, allows us to define a similitude D_{x}, with ratio $\rho = \rho(\mathrm{x})$, by:

$$D_{\mathrm{x}} : X \mapsto \mathrm{x} = \phi X \phi^\dagger. \tag{1.279}$$

And the components D_μ^ν of the four vectors D_μ are the terms of the matrix of this similitude D_{x} because:

$$\mathrm{x} = \mathrm{x}^\mu\sigma_\mu = \phi X^\nu\sigma_\nu\phi^\dagger = X^\nu\phi\sigma_\nu\phi^\dagger = X^\nu\mathrm{D}_\nu = X^\nu\mathrm{D}_\nu^\mu\sigma_\mu \ ; \quad \mathrm{x}^\mu = \mathrm{D}_\nu^\mu X^\nu. \tag{1.280}$$

There is no difference between the $M'M$ product giving the composition $R'\circ R$ of the similitudes, and the product $M\phi$ which gives the transformation of the wave under a similitude, and which then induces a composition of the similitudes $\mathrm{D}'_{\mathrm{x}'} = R\circ \mathrm{D}_{\mathrm{x}}$:

$$\mathrm{x}' = M\mathrm{x}M^\dagger = M\phi X\phi^\dagger M^\dagger = (M\phi)X(M\phi)^\dagger = \phi' X\phi'^\dagger \tag{1.281}$$

This implies that the X introduced in (1.279) does not change when seen by the observer at x or by the observer at x′. It is independent of the observer. We may remark that the dinum of X is null, because the dinum of x is 1 while the dinum of ϕ and of $\phi^\dagger$ are 1/2.

We may then name the manifold of X the **invariant space-time**.

1.9 Energy–momentum, Lorentz force

The calculation of the Lorentz force may be done in space-time algebra; this needs the use of the method of calculation explained in [79] and [83]. Since the wave of the electron only has value in the even subalgebra and since this even subalgebra is isomorphic to the Cl_3 algebra, we are able to calculate the Lorentz force easier using Cl_3. First question: what energy–momentum density may be attached to the Dirac wave? Quantum field theory derives this density from the Lagrangian density, and the invariance of the Lagrangian density under space-time translation allows us to define a tensor density of energy–momentum from Noether's theorem. But the Lagrangian density is also a problem: several textbooks, old [2][105] or new like Wikipedia, give this Lagrangian density as:

$$\mathcal{L} = i\overline{\psi}\partial\psi - m\overline{\psi}\psi. \tag{1.282}$$

Besides the error in sign, because they wrongly indicate a negative energy density, these authors are apparently unaware of the complex character of this Lagrangian density. Some other authors are more precise [96] and give the Lagrangian density as:

$$\mathcal{L} = \Re(-i\overline{\psi}\partial\psi) + m\overline{\psi}\psi. \tag{1.283}$$

This is the form that we previously used. We must recall that passing from this form to the improved equation replaces $\overline{\psi}\psi = \rho\cos(\beta)$ with ρ. It is interesting to see why the absence of rigor in this part of the theory has no impact on the studies that came after and remains unnoticed. We have obtained this Lagrangian density in (1.133) as the real part of the wave (because the real field is always included in real Clifford algebras). Since we work in the Pauli algebra, this is the real part (in the complex field) of the trace of the matrix. This trace also has an imaginary part which gives the conservation of the probability current ($\partial_\mu \mathrm{J}^\mu = 0$). This explains why the Lagrangian density of the seemingly complex form (1.283) is nevertheless correct, the imaginary part being automatically null. Moreover the formula is shorter and more convenient for an introduction. The physical reason explaining why the two formulas for $\mathcal{L}$ give the same result is the invariance under the electric gauge, associated with the conservation of the probability current by Noether's theorem. We may then begin all calculations from (1.283), using complex variables to calculate densities which will nevertheless all be real as a result of the electric gauge invariance.

Like the Lagrangian density, the tensor T is the sum of a left part depending only on the left wave η and of a right part depending only on the right wave ξ. We will again see in Chapter 2 this important partition between right and left waves, a partition which is invariant under Cl_3^*, thus relativistically invariant. The notion that these left and right waves are the fundamental fields was obtained and used by G. Lochak [84]–[90] for his theory of the magnetic monopole. This partition is actually important for all tensor densities that we obtain from the spinors. The existence of a energy–momentum tensor T_R for the right waves and another tensor T_L for the left waves implies the existence of **two tensors of energy–momentum**, the tensor $T = T_R + T_L$ and the tensor $V = T_L - T_R$ noted by O. Costa de Beauregard [51] as not yet classically interpreted. This tensor V is simply the difference between the right and left tensors [32]. The Tétrode tensor becomes:

$$T^\mu_\nu = \frac{m}{k\mathbf{l}}T^\mu_{L\nu} + \frac{m}{k\mathbf{l}}T^\mu_{R\nu},$$

$$T^\mu_{L\nu} = \frac{i}{2}\Big[-\eta^{1\dagger}\sigma^\mu\partial_\nu\eta^1 + (\partial_\nu\eta^{1\dagger})\sigma^\mu\eta^1\Big] + (qA_\nu + \mathbf{l}\mathrm{v}_\nu)\mathrm{D}^{1\mu}_L, \tag{1.284}$$

$$T^\mu_{R\nu} = \frac{i}{2}\Big[-\xi^{1\dagger}\widehat{\sigma}^\mu\partial_\nu\xi^1 + (\partial_\nu\xi^{1\dagger})\widehat{\sigma}^\mu\xi^1\Big] + (qA_\nu + \mathbf{r}\mathrm{v}_\nu)\mathrm{D}^{1\mu}_R. \tag{1.285}$$

In space-time algebra, the energy–momentum tensor $T(\mathrm{u}) = T(\mathrm{u},\mathrm{x})$ is interpreted by Hestenes [75] as the flux of energy–momentum through a hypersurface with normal vector u at the space-time point x. It is a vectorial function of a vectorial variable [33]:

$$T(\mathrm{u}) = T(\mathrm{u}_\mu\sigma^\mu) = \mathrm{u}_\mu T(\sigma^\mu). \tag{1.286}$$

Thus the tensor is completely defined by the vectors

$$T^\mu = T(\sigma^\mu), \tag{1.287}$$

which satisfy

$$T^\mu = T^\mu_\nu\sigma^\nu \ ; \ \ T^\mu_\nu = T^\mu\cdot\sigma_\nu. \tag{1.288}$$

We then get:

$$\partial_\mu T^\mu = (\partial_\mu T^\mu_\nu)\sigma^\nu, \tag{1.289}$$

$$\partial_\mu T^\mu_{L\nu} = \frac{i}{2}\partial_\mu[-\eta^{1\dagger}\sigma^\mu\partial_\nu\eta^1 + (\partial_\nu\eta^{1\dagger})\sigma^\mu\eta^1] + \partial_\mu[(qA_\nu + \mathbf{l}\mathrm{v}_\nu)\mathrm{D}^{1\mu}_L], \tag{1.290}$$

$$\partial_\mu T^\mu_{R\nu} = \frac{i}{2}\partial_\mu[-\xi^{1\dagger}\widehat{\sigma}^\mu\partial_\nu\xi^1 + (\partial_\nu\xi^{1\dagger})\widehat{\sigma}^\mu\xi^1] + \partial_\mu[(qA_\nu + \mathbf{r}\mathrm{v}_\nu)\mathrm{D}^{1\mu}_R], \tag{1.291}$$

where the partial derivatives commute and the $\mathbf{J}$ current is conservative. We then get:

$$\partial_\mu[(qA_\nu + \mathbf{l}\mathrm{v}_\nu)\mathrm{D}^{1\mu}_L] = (q\partial_\mu A_\nu + \mathbf{l}\partial_\mu\mathrm{v}_\nu)\mathrm{D}^{1\mu}_L + (qA_\nu + \mathbf{l}\mathrm{v}_\nu)\partial_\mu\mathrm{D}^{1\mu}_L$$
$$= (q\partial_\mu A_\nu + \mathbf{l}\partial_\mu\mathrm{v}_\nu)\mathrm{D}^{1\mu}_L \tag{1.292}$$

$$\partial_\mu[(qA_\nu + \mathbf{r}\mathrm{v}_\nu)\mathrm{D}^{1\mu}_R] = (q\partial_\mu A_\nu + \mathbf{r}\partial_\mu\mathrm{v}_\nu)\mathrm{D}^{1\mu}_R + (qA_\nu + \mathbf{r}\mathrm{v}_\nu)\partial_\mu\mathrm{D}^{1\mu}_R$$
$$= (q\partial_\mu A_\nu + \mathbf{r}\partial_\mu\mathrm{v}_\nu)\mathrm{D}^{1\mu}_R \tag{1.293}$$

32. To derive the dynamics of the electron Hestenes started from a different tensor [75] containing only the differential terms. Yet he added the electromagnetic part, his tensor is thus identical to ours. Hestenes' calculation is complicated by failing to distinguish the right and left parts of the wave. It is only with a convenient choice of the γ_μ matrices that we can easily see the tensors as sums of a left and a right part.

33. For GR this is an important point that comes from quantum physics: vectors are the only useful tensors.

$$\partial_\mu \mathrm{T}^\mu_{L\nu} = \frac{i}{2}[-(\nabla\eta^1)^\dagger\partial_\nu\eta^1 - \eta^{1\dagger}\partial_\nu(\nabla\eta^1) + \partial_\nu(\nabla\eta^1)^\dagger\eta^1 + \partial_\nu\eta^{1\dagger}\nabla\eta^1] + (q\partial_\mu A_\nu + \mathbf{l}\partial_\mu \mathrm{v}_\nu)\mathrm{D}^{1\mu}_L, \tag{1.294}$$

$$\partial_\mu \mathrm{T}^\mu_{R\nu} = \frac{i}{2}[-(\widehat{\nabla}\xi^1)^\dagger\partial_\nu\xi^1 - \eta^{1\dagger}\partial_\nu(\widehat{\nabla}\xi^1) + \partial_\nu(\widehat{\nabla}\xi^1)^\dagger\xi^1 + \partial_\nu\xi^{1\dagger}\widehat{\nabla}\xi^1 + (q\partial_\mu A_\nu + \mathbf{r}\partial_\mu \mathrm{v}_\nu)\mathrm{D}^{1\mu}_R. \tag{1.295}$$

Next we use the wave equations of η^1 and ξ^1 which are equivalent to the system:

$$\nabla\eta^1 = -i(qA + \mathbf{l}\mathrm{v})\eta^1;\ (\nabla\eta^1)^\dagger = i\eta^{1\dagger}(qA + \mathbf{l}\mathrm{v}),$$
$$\widehat{\nabla}\xi^1 = -i(q\widehat{A} + \mathbf{r}\widehat{\mathrm{v}})\xi^1;\ (\widehat{\nabla}\xi^1)^\dagger = i\xi^{1\dagger}(q\widehat{A} + \mathbf{r}\widehat{\mathrm{v}}). \tag{1.296}$$

We thus obtain:

$$\partial_\mu T^\mu_{L\nu} = [q(\partial_\mu A_\nu - \partial_\nu A_\mu) + \mathbf{l}(\partial_\mu \mathrm{v}_\nu - \partial_\nu \mathrm{v}_\mu)]\mathrm{D}^{1\mu}_L, \tag{1.297}$$
$$\partial_\mu T^\mu_{R\nu} = [q(\partial_\mu A_\nu - \partial_\nu A_\mu) + \mathbf{r}(\partial_\mu \mathrm{v}_\nu - \partial_\nu \mathrm{v}_\mu)]\mathrm{D}^{1\mu}_R, \tag{1.298}$$
$$k\partial_\mu T^\mu_\nu = q(\partial_\mu A_\nu - \partial_\nu A_\mu)\Big(\frac{m}{\mathbf{l}}\mathrm{D}^{1\mu}_L + \frac{m}{\mathbf{r}}\mathrm{D}^{1\mu}_R\Big) + m(\partial_\mu \mathrm{v}_\nu - \partial_\nu \mathrm{v}_\mu)\mathrm{J}^\mu. \tag{1.299}$$

The electromagnetic field F et and the gravitational field G are defined as

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu;\ G_{\mu\nu} := \partial_\mu \mathrm{v}_\nu - \partial_\nu \mathrm{v}_\mu. \tag{1.300}$$

We hence obtain:

$$\partial_\mu \mathrm{T}^\mu_\nu = qF_{\mu\nu}\underline{\mathrm{J}}^\mu + mG_{\mu\nu}\mathrm{J}^\mu, \tag{1.301}$$
$$\underline{\mathrm{J}} := \frac{m}{\mathbf{l}}\mathrm{D}^1_L + \frac{m}{\mathbf{r}}\mathrm{D}^1_R. \tag{1.302}$$

If $m \approx \mathbf{l} \approx \mathbf{r}$ and with the total field

$$\mathbf{F} := F + \frac{m}{q}G, \tag{1.303}$$

we obtain:

$$\partial_\mu T^\mu \approx \Big(F_{\mu\nu} + \frac{m}{q}G_{\mu\nu}\Big)q\mathrm{J}^\mu\sigma^\nu = \mathbf{F}_{\mu\nu}q\mathrm{J}^\mu\sigma^\nu. \tag{1.304}$$

We then obtain a Lorentz relation for a space-time vector j ($\mathrm{j}^0 = \rho_e$: density of charge; $\vec{\mathrm{j}}$: density of electric current) under the relativistic form:

$$\mathrm{j} = q\mathrm{J};\ \partial_\mu T^\mu = \mathbf{F}_{\mu\nu}\mathrm{j}^\mu\sigma^\nu. \tag{1.305}$$

Thus with:

$$\mathbf{F} = \vec{E} + i\vec{H}$$
$$\mathrm{j} = q\mathrm{J} = \rho_e + \vec{\mathrm{j}};\ \mathrm{f} = \mathrm{f}_0 + \vec{\mathrm{f}}, \tag{1.306}$$

where $\vec{E}$ is the electric field, $\vec{H}$ is the magnetic field, ρ_e is the electric charge, $\vec{\mathrm{j}}$ is the density of electric current and $\vec{\mathrm{f}}$ is the force density, (1.305) is equivalent to:

$$\vec{\mathrm{f}} = \rho_e \vec{E} + \vec{\mathrm{j}} \times \vec{H};\ \mathrm{f}_0 = \vec{E} \cdot \vec{\mathrm{j}}. \tag{1.307}$$

This is obviously very important to unify the laws of physics: except the Lorentz force that we simply obtained as a consequence of the Dirac equation or of our improved equation, only the gravitational field is able to yield the gravitational force as a consequence of field equations. This means that the Standard Model, in the fully relativistic manner used here, is as good as general relativity to obtain the motion of field sources.

1.10 Electromagnetic field

We recall that the electric current $\mathrm{j} = q\mathbf{J}$ is linked to the chiral currents $\mathrm{D}_L = L^1 L^{1\dagger}$, $\mathrm{D}_R = R^1 R^{1\dagger}$. Without the magnetic monopole, all of Maxwell's laws are reduced to (see A.3.6):

$$F = \nabla \widehat{A};\ \widehat{F} = \widehat{\nabla} A;\ \nabla \widehat{F} = \mathrm{j}, \tag{1.308}$$

and thus at the second order:

$$\nabla(\widehat{\nabla} A) = (\nabla \widehat{\nabla}) A = \Box A = \mathrm{j} = q\mathbf{J} = \frac{e}{\hbar c}\Big(\frac{m}{k\mathbf{l}}\mathrm{D}^1_L + \frac{m}{k\mathbf{r}}\mathrm{D}^1_R\Big). \tag{1.309}$$

Since $\mathbf{J}$ is a linear combination of chiral currents, we study two fields:

$$F_L := \nabla \widehat{\mathrm{D}}^1_L;\ \widehat{F}_R := \widehat{\nabla} \mathrm{D}^1_R. \tag{1.310}$$

The left field F_L satisfies

$$\begin{aligned} F_L = \vec{E}_L + i\vec{H}_L &= (\partial_0 - \vec{\partial})(\mathrm{D}^{10}_L - \vec{\mathrm{D}}^1_L) \\ &= \partial_\mu \mathrm{D}^{1\mu}_L - \partial_0 \vec{\mathrm{D}}^1_L - \vec{\partial}\mathrm{D}^{10}_L + i\vec{\partial} \times \vec{\mathrm{D}}^1_L \end{aligned} \tag{1.311}$$

and we thus obtain:

$$0 = \partial_\mu \mathrm{D}^{1\mu}_L \tag{1.312}$$

$$\vec{E}_L = -\partial_0 \vec{\mathrm{D}}^1_L - \vec{\partial}\mathrm{D}^{10}_L;\ \vec{H}_L = \vec{\partial} \times \vec{\mathrm{D}}^1_L. \tag{1.313}$$

The right field F_R satisfies

$$\begin{aligned} \widehat{F}_R = -\vec{E}_R + i\vec{H}_R &= (\partial_0 + \vec{\partial})(\mathrm{D}^{10}_R + \vec{\mathrm{D}}^1_R) \\ &= \partial_\mu \mathrm{D}^{1\mu}_R + \partial_0 \vec{\mathrm{D}}^1_R + \vec{\partial}\mathrm{D}^{10}_R + i\vec{\partial} \times \vec{\mathrm{D}}^1_R \end{aligned} \tag{1.314}$$

and we thus obtain:

$$0 = \partial_\mu \mathrm{D}^{1\mu}_R \tag{1.315}$$

$$\vec{E}_R = -\partial_0 \vec{\mathrm{D}}^1_R - \vec{\partial}\mathrm{D}^{10}_R;\ \vec{H}_R = \vec{\partial} \times \vec{\mathrm{D}}^1_R. \tag{1.316}$$

We express the covariant derivatives in (1.198) as:

$$d^L_\mu = -i\partial_\mu + l_\mu;\ l_\mu := qA_\mu + \mathbf{l}\mathrm{v}_\mu,$$
$$d^R_\mu = -i\partial_\mu + r_\mu;\ r_\mu := qA_\mu + \mathbf{r}\mathrm{v}_\mu. \tag{1.317}$$

The wave equations for η^1 and ξ^1 can be expressed as:

$$\sigma^\mu\partial_\mu\eta^1 = -i\sigma^\mu l_\mu\eta^1, \tag{1.318}$$
$$\widehat{\sigma}^\mu\partial_\mu\xi^1 = -i\widehat{\sigma}^\mu r_\mu\xi^1. \tag{1.319}$$

Densities which are components of the energy–momentum tensor may be separated into two parts:

$$\begin{aligned} T^\mu_{L\nu} &= \frac{1}{2}\Big[\eta^{1\dagger}\sigma^\mu(-i\partial_\nu\eta^1 + l_\nu\eta^1) + (i\partial_\nu\eta^{1\dagger} + l_\nu\eta^{1\dagger})\sigma^\mu\eta^1\Big] \\ &= \frac{i}{2}\Big[-\eta^{1\dagger}\sigma^\mu\partial_\nu\eta^1 + (\partial_\nu\eta^{1\dagger})\sigma^\mu\eta^1\Big] + l_\nu\eta^{1\dagger}\sigma^\mu\eta^1, \end{aligned} \tag{1.320}$$

$$\begin{aligned} T^\mu_{R\nu} &= \frac{1}{2}\Big[\xi^{1\dagger}\widehat{\sigma}^\mu(-i\partial_\nu\xi^1 + r_\nu\xi^1) + (i\partial_\nu\xi^{1\dagger} + r_\nu\xi^{1\dagger})\widehat{\sigma}^\mu\xi^1\Big] \\ &= \frac{i}{2}\Big[-\xi^{1\dagger}\widehat{\sigma}^\mu\partial_\nu\xi^1 + (\partial_\nu\xi^{1\dagger})\widehat{\sigma}^\mu\xi^1\Big] + r_\nu\xi^{1\dagger}\widehat{\sigma}^\mu\xi^1. \end{aligned} \tag{1.321}$$

The equation of the left wave gives:

$$\partial_0\eta^1 + \sigma^1\partial_1\eta^1 = \sigma_2\partial_2\eta^1 + \sigma_3\partial_3\eta^1 - i(l_0 + l_1\sigma^1 + l_2\sigma^2 + l_3\sigma^3)\eta^1. \tag{1.322}$$

Multiplying on the left side by $\eta^{1\dagger}\sigma^1$ we obtain:

$$\begin{aligned} &\eta^{1\dagger}\sigma^1\partial_0\eta^1 + \eta^{1\dagger}\partial_1\eta^1 \\ &= -i\eta^{1\dagger}\sigma^3\partial_2\eta^1 + i\eta^{1\dagger}\sigma^2\partial_3\eta^1 - i\eta^{1\dagger}(l_0\sigma^1 + l_1 + i\sigma^3 l_2 - i\sigma^2 l_3)\eta^1. \end{aligned} \tag{1.323}$$

Using the adjoint, and then adding, we obtain:

$$\begin{aligned} &(\partial_0\eta^{1\dagger})\sigma^1\eta^1 + (\partial_1\eta^{1\dagger})\eta^1 \\ &= i(\partial_2\eta^{1\dagger})\sigma^3\eta^1 - i(\partial_3\eta^{1\dagger})\sigma^2\eta^1 + i\eta^{1\dagger}(\sigma^1 l_0 + l_1 - i\sigma^3 l_2 + i\sigma^2 l_3)\eta^1, \end{aligned} \tag{1.324}$$

$$\begin{aligned} &\partial_0(\eta^{1\dagger}\sigma^1\eta^1) + \partial_1(\eta^{1\dagger}\sigma^0\eta^1) \\ &= -i\eta^{1\dagger}\sigma^3\partial_2\eta^1 + i(\partial_2\eta^{1\dagger})\sigma^3\eta^1 + 2l_2\eta^{1\dagger}\sigma^3\eta^1 \\ &+ i\eta^{1\dagger}\sigma^2\partial_3\eta^1 - i(\partial_3\eta^{1\dagger})\sigma^2\eta^1 - 2l_3\eta^{1\dagger}\sigma^2\eta^1. \end{aligned} \tag{1.325}$$

We thus get:[34]

$$\partial_0\mathrm{D}^{11}_L + \partial_1\mathrm{D}^{10}_L = 2T^3_{L2} - 2T^2_{L3},$$
$$-E^1_L = 2(T^3_{L2} - T^2_{L3}). \tag{1.326}$$

34. We must notice that this calculation is completely dependent on the number of space dimensions: three. It is linked to the existence of the cross product and mixed product, with the dimension $1+3+3+1$ of the Cl_3 algebra, which gives $\sigma_1\sigma_2\sigma_3 = i$, and so on.

Circularly permuting indices, we have:

$$\begin{aligned} E_L^1 &= 2(T_{L3}^2 - T_{L2}^3), \\ E_L^2 &= 2(T_{L1}^3 - T_{L3}^1), \\ E_L^3 &= 2(T_{L2}^1 - T_{L1}^2). \end{aligned} \tag{1.327}$$

Now subtracting (1.323)−(1.324) we obtain:

$$\begin{aligned} &\eta^{1\dagger}\sigma^1\partial_0\eta^1 - (\partial_0\eta^{1\dagger})\sigma^1\eta^1 + 2il_0\eta^{1\dagger}\sigma^1\eta^1 \\ &+ \eta^{1\dagger}\sigma^0\partial_1\eta^1 - (\partial_1\eta^{1\dagger})\sigma^0\eta^1 + 2il_1\eta^{1\dagger}\sigma^0\eta^1 \\ &= -i\partial_2(\eta^{1\dagger}\sigma^3\eta^1) + i\partial_3(\eta^{1\dagger}\sigma^2\eta^1). \end{aligned} \tag{1.328}$$

Next dividing by i and permuting indices, we obtain:

$$\begin{aligned} H_L^1 &= -2(T_{L0}^1 + T_{L1}^0), \\ H_L^2 &= -2(T_{L0}^2 + T_{L2}^0), \\ H_L^3 &= -2(T_{L0}^3 + T_{L3}^0). \end{aligned} \tag{1.329}$$

The strong link obtained here between the electromagnetic field and the energy–momentum tensor of the electron wave is thus proper to three-dimensional space.[35]

The equation of right waves gives:

$$\partial_0\xi^1 + \sigma_1\partial_1\xi^1 = \sigma^2\partial_2\xi^1 + \sigma^3\partial_3\xi^1 - i(r_0 + r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3)\xi^1. \tag{1.330}$$

Multiplying on the left side by $\xi^{1\dagger}\sigma_1$ we obtain:

$$\begin{aligned} \xi^{1\dagger}\widehat{\sigma}^1\partial_0\xi^1 + \xi^{1\dagger}\widehat{\sigma}^0\partial_1\xi^1 &= i\xi^{1\dagger}\widehat{\sigma}^3\partial_2\xi^1 - i\xi^{1\dagger}\widehat{\sigma}^2\partial_3\xi^1 \\ &\quad - ir_0\mathrm{D}_R^{11} - ir_1\mathrm{D}_R^{10} - r_2\mathrm{D}_R^{13} + r_3\mathrm{D}_R^{12}. \end{aligned} \tag{1.331}$$

Next using the adjoint and adding, this gives:

$$\partial_0\mathrm{D}_R^{11} + \partial_1\mathrm{D}_R^{10} = -2T_{R2}^3 + 2T_{R3}^2, \tag{1.332}$$

$$\begin{aligned} E_R^1 &= 2(-T_{R3}^2 + T_{R2}^3), \\ E_R^2 &= 2(-T_{R1}^3 + T_{R3}^1), \\ E_R^3 &= 2(-T_{R2}^1 + T_{R1}^2). \end{aligned} \tag{1.333}$$

While by subtracting we obtain:

$$\begin{aligned} H_R^1 &= 2(T_{R0}^1 + T_{R1}^0), \\ H_R^2 &= 2(T_{R0}^2 + T_{R2}^0), \\ H_R^3 &= 2(T_{R0}^3 + T_{R3}^0). \end{aligned} \tag{1.334}$$

35. This is a sufficient reason to follow Baylis [2] and prefer Cl_3 to $Cl_{1,3}$, the space-time algebra previously used by many physicists like Hestenes, Boudet and Lasenby. None of them got relations (1.329).

It is possible to gather together the left and right parts of the electromagnetic field by using Costa de Beauregard's tensor V such that:

$$V^\mu_\nu := 2(T^\mu_{L\nu} - T^\mu_{R\nu}), \tag{1.335}$$

because we obtain:

$$\begin{aligned} E^1 &= V^2_3 - V^3_2;\ H^1 = -V^1_0 - V^0_1, \\ E^2 &= V^3_1 - V^1_3;\ H^2 = -V^2_0 - V^0_2, \\ E^3 &= V^1_2 - V^2_1;\ H^3 = -V^3_0 - V^0_3. \end{aligned} \tag{1.336}$$

The electromagnetic field is thus naturally the difference of two chiral fields, and the polarization of light is the direct consequence of the structure of the material wave of the electron. Moreover a difference exists between the two parts, even in the second-order equation coming from Maxwell's laws. To obtain the D'Alembertian of the potentials, $\widehat{\nabla}\nabla\widehat{\mathrm{D}}_L$ is used on the left part of the electromagnetic field, while $\nabla\widehat{\nabla}\mathrm{D}_R$ is used on the right part. The electromagnetic field is a pure bivector field ($F = \vec{E} + i\vec{H}$) with neither scalar nor pseudoscalar part: this comes from the nature of F as the gradient of a vector, without a pseudovector part (this will change in the second chapter), and from the conservation of D_L and D_R currents.

The potential A is not only a mathematical tool for the calculation of the electromagnetic field, it has a kind of physical reality; this was claimed by O. Costa de Beauregard [52], following L. de Broglie [55] [56]. Still more important, the potential A is not exterior to the wave, but totally dependent on the wave, which is necessary in any true theory of fields. We will explain in the following chapter why A seems exterior.

The wave equation of the electron comes from a Lagrangian mechanism, and we will explain in the next Chapter exactly how and why. And we do not need a Lagrangian of the electromagnetic field. This field is entirely incorporated into the quantum field of the electron. Maxwell's laws and the Lorentz force are necessary consequences of the improved wave equation, for the densities of energy–momentum and of kinetic momentum,. The electromagnetic field does not need a Lagrangian density and its associated energy–momentum, because the electromagnetic field itself is energy–momentum. This is directly linked, in light, to the existence, claimed by Einstein [64], of **quanta of energy–momentum**, nowadays called photons,.

1.11 Absolute length and time units

This study earlier introduced in 1.5.5 a constant k which has dimension T^2/M. This constant may be linked to the gravitational constant G with the following definition of l_a, using the fine structure constant α:

$$e^2 = \alpha\hbar c;\ l_a^2 := \frac{Ge^2}{c^4} = \alpha\frac{G\hbar}{c^3} = \alpha l_P^2, \tag{1.337}$$

where l_P is the Planck length. We name $l_a = \sqrt{\alpha} l_P$ the absolute length. The inclusion of the constant α in the natural units of the Planck system was introduced by one of us in [102]. Thus $\alpha\hbar/c$ which has dimension ML may be set as $m_a l_a$, where m_a is the absolute unit of mass, which gives:

$$m_a = \alpha\frac{\hbar}{c}\sqrt{\frac{c^3}{\alpha G\hbar}} = \sqrt{\alpha}\sqrt{\frac{\hbar c}{G}} = \sqrt{\alpha} m_P \tag{1.338}$$

where m_P is the Planck mass. We may also use an absolute time unit t_a and an absolute constant k:

$$t_a := \frac{l_a}{c} = \sqrt{\alpha} t_P;\ k := \frac{t_a^2}{m_a} = \frac{l_a^2}{m_a c^2} = \frac{l_a^2}{E_a}, \tag{1.339}$$

naming E_a the absolute unit of energy. As the main uncertainty of measurement comes from G ($G = 6.67430(15) \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$), we obtain:

$$l_a = 1.38068(3) \times 10^{-36}\mathrm{m}, \tag{1.340}$$

$$m_{abs} = 1.85921(4) \times 10^{-9}\mathrm{kg}, \tag{1.341}$$

$$t_a = 4.60545(10) \times 10^{-45}\mathrm{s}, \tag{1.342}$$

$$k = 1.140815(25) \times 10^{-80}\mathrm{s}^2\mathrm{kg}^{-1}, \tag{1.343}$$

$$\frac{1}{k} = 8,76566(19) \times 10^{79}\mathrm{kg}\,\mathrm{s}^{-2}. \tag{1.344}$$

Hence we obtain for the $\mathbf{J}$ current:

$$\frac{\mathbf{J}}{\hbar c} = \frac{1}{k\hbar c}\Big(\frac{m}{\mathbf{l}}\mathrm{D}_L + \frac{m}{\mathbf{r}}\mathrm{D}_R\Big) = \frac{\alpha}{l_a^3}\Big(\frac{m}{\mathbf{l}}\mathrm{D}_L + \frac{m}{\mathbf{r}}\mathrm{D}_R\Big). \tag{1.345}$$

Thus $\frac{\mathbf{J}}{\hbar c}$ has the dimension L^{-3} of a probability density and the tensors T have the dimension of energy densities. Hence it is the same with the electromagnetic field.

Chapter 2

Weak Interactions (Lepton case)

We use the Clifford algebra $Cl_{3,3}$ for the waves of all fermions and antifermions of the first generation. This includes a magnetic monopole that is also the complete neutrino, with both left and right waves. We study the new tensor densities that come from the extension of the electron wave. We transpose to Clifford algebra the covariant derivative of the electroweak gauge group. This covariant derivative is compatible with the mass term of the improved wave equation. We generalize to the lepton wave the Lagrangian density of the electron as well as its double link with the gauge fields. The recursion of the wave equation allows us to obtain the properties of the gauge fields. The lepton wave equation is form-invariant under Cl_3^*. The gauge invariance group is the $U(1) \times SU(2)$ group of the electroweak interactions. We also obtain the value of each charge of particle and antiparticle. The constraints imposed by this gauge group allow us to calculate the gauge potentials and to simplify the wave equation. The particular case of the electron fixes the value of the Weinberg–Salam angle to $30°$. We study the energy–momentum tensor density. We obtain the Lorentz force. We derive the dynamics of the magnetic monopole. We study the kinetic momentum tensor density and we derive the quantization of the kinetic momentum from the form-invariance of the kinetic momentum under Cl_3^*.

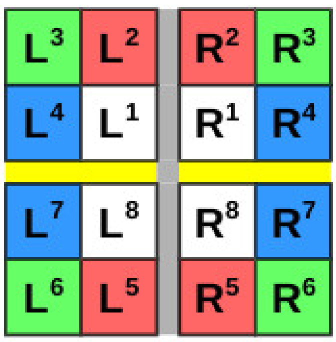
2.1 From the electron wave to the complete wave

In the first chapter we ascribed a dinum of 1/2 to the wave of the electron, and we saw that the electromagnetic field had a dinum of 0. Thus some quantities have a dinum while others do not. The origin of the concept of a dinum is relativistic quantum physics. Since QFT replaces the electromagnetic field with a field of creator and annihilator operators, we thus postulate this fermion field as a field of operators:

$$\Psi : \phi \mapsto \phi_e,\ \ \phi_e = \Psi(\phi); \Psi \in \mathrm{End}(Cl_3);\ \ \mathrm{x} = \phi_e X \widetilde{\phi}_e. \tag{2.1}$$

where X belongs to the self-adjoint part of Cl_3, and ϕ_e is the wave of the electron. This is the fourth major change that we introduce: **space-time is not a starting point, but the consequence of the fermion field value**. This will be essential to incorporating gravitation into the Theory of Everything (see Chapter 4). The $\mathrm{x} = \mathrm{x}^\mu \sigma_\mu \in Cl_3$ is the general linear element of space-time in special relativity, and the general element of any tangent space-time in general relativity. Time is the fourth component: $\mathrm{x}^0 = ct$. We do not need additional dimensions for the space-time manifold.

For the first generation of fundamental fermions the SM accounts for 16 fermions: eight particles and their corresponding antiparticles. We have just studied the case of the electron and its antiparticle the positron. These objects are not the only ones in the Standard Model. They are only examples of what are called *fermions*. In ordinary matter other fermions exist within the atoms, whose nuclei are made of protons and neutrons that are themselves composed of colored quarks. Besides an electron and its neutrino or their equivalent particles, this ordinary matter forms what is called the "first generation." Each generation includes two quarks with three color states each. Thus we get eight waves similar to the electron wave ϕ_e. We label these waves from one to eight. Each one of these eight waves so labeled has a left part and a right part. Here we study the general case, while in [28] we simplified the study by neglecting the right waves of the quarks. In [47] we summarized the scene in the following picture:



The quarks of the first generation are called the up (u) and down (d) quarks, and the couple $d-u$ is similar to $n-e$ in electroweak interactions, but with differences since the electric charge of the u quark will be $\frac{2}{3}|e|$, while the charge of the d quark will be $-\frac{1}{3}|e|$. For the lepton sector of each generation, the charges of the antiparticles appear opposite to the charges of particles. As we saw in Chapter 1, neither the charges nor the mass actually change. But the wave equations change because all partial derivatives change sign, and the right and left parts of the waves are exchanged. Without this identity between the wave of the particle and the wave of the antiparticle we should count not only 64 parameters but 128, and End(Cl_3) offers only 64 dimensions. The three "color charges" are called r, g, b (red, green, blue). The lepton wave that we study in this chapter is the white one at the center of the diagram. The quark waves are placed at the colored perimeter of the diagram; we will study them in the next chapter. This diagram indicates two symmetries that are both left–right symmetries: we placed the left waves on the left side and the right waves on the right side. We recall that the invariance group acts differently on right R^n and left L^n waves – this is precisely the source of the symmetry. The Cl_3^* invariance group is also the source of the second symmetry between the upper part of the diagram, on which the action is a multiplication on the left side, while the action on the lower part is a multiplication on the right side [1]. This second symmetry exchanges, for instance, the four red cases: those of the upper part containing the waves L^2 and R^2 of the d quark with color r, and those of the lower part containing the waves L^5 and R^5 of the u quark with color r. This double symmetry is well known in the framework of Lie groups and Lie algebras: the $GL(n,\mathbb{C})$ groups have four kinds of representations. In these symmetries quarks and leptons are highly similar.

We now see what differentiates those in the perimeter of the diagram from those at the center. For each quarter of the diagram we have one white box and three colored boxes; thus the whole wave of the first generation also comes from a mathematical object linked to Cl_3 since it takes value in the algebra of all endomorphisms on this linear space: End(Cl_3). It also happens that this ring is the Clifford algebra $Cl_{3,3}$ (we study this algebra in B.2). This algebra is a 64-dimensional linear space on the $\mathbb{R}$ field. Therefore we will use the function Ψ, with value in $Cl_{3,3}$, as quantum wave of second quantization. This algebra contains eight supplementary linear spaces similar to Cl_3. So we will use these eight linear spaces to obtain eight waves linearly similar to the wave of the electron in Chapter 1. With (B.95) we have

$$\Psi = \Psi(\mathrm{x}) = \begin{pmatrix} \Psi_l + i\Psi_b & \Psi_r + \Psi_g \\ \Psi_r - \Psi_g & \Psi_l - i\Psi_b \end{pmatrix}, \tag{2.2}$$

1. This symmetry that inverts the order of all products is called reversion in Clifford algebra (see A.1).

$$\Psi_l = \mathcal{P}_1 - i\mathcal{I}_1;\ \mathcal{P}_1 = \begin{pmatrix} \phi_e & 0 \\ 0 & \widehat{\phi}_e \end{pmatrix};\ \mathcal{I}_1 = \begin{pmatrix} 0 & \phi_n^\dagger \\ \overline{\phi}_n & 0 \end{pmatrix}, \tag{2.3}$$

$$\Psi_r = -i\mathcal{P}_2 + \mathcal{I}_2;\ \mathcal{P}_2 = \begin{pmatrix} \phi_{dr} & 0 \\ 0 & \widehat{\phi}_{dr} \end{pmatrix};\ \mathcal{I}_2 = \begin{pmatrix} 0 & \phi_{ur}^\dagger \\ \overline{\phi}_{ur} & 0 \end{pmatrix}, \tag{2.4}$$

$$\Psi_g = -i\mathcal{P}_3 + \mathcal{I}_3;\ \mathcal{P}_3 = \begin{pmatrix} \phi_{dg} & 0 \\ 0 & \widehat{\phi}_{dg} \end{pmatrix};\ \mathcal{I}_3 = \begin{pmatrix} 0 & \phi_{ug}^\dagger \\ \overline{\phi}_{ug} & 0 \end{pmatrix}, \tag{2.5}$$

$$\Psi_b = -i\mathcal{P}_4 + \mathcal{I}_4;\ \mathcal{P}_4 = \begin{pmatrix} \phi_{db} & 0 \\ 0 & \widehat{\phi}_{db} \end{pmatrix};\ \mathcal{I}_4 = \begin{pmatrix} 0 & \phi_{ub}^\dagger \\ \overline{\phi}_{ub} & 0 \end{pmatrix}. \tag{2.6}$$

The Ψ term is then composed of two different kinds of terms: Ψ_l which is a single term, and Ψ_r, Ψ_g and Ψ_b, which are three similar terms, all different from Ψ_l. This means that we distinguish between a lepton part Ψ_l and a quark part $(\Psi_r,\ \Psi_g,\ \Psi_b)$ directly from the definition of the whole quantum wave.

In this chapter we study the Ψ_l wave, which is a function of space-time in $Cl_{3,1}$. And since i is the 3-vector term of Cl_3, which commutes with any term in Cl_3, when we restrict Ψ_l to its first row containing the 1 and 8 indices we may consider a function with value in $Cl_{3,1}$ as a function in $Cl_3 \times Cl_3$:

$$\Psi_l = (\phi_e \quad -i\phi_n^\dagger) = (\phi^1 \quad \phi^{8\dagger}) \in Cl_3 \times Cl_3. \tag{2.7}$$

The Ψ_l wave is made of two similar waves, $\phi_e = \phi^1$ which is, in the picture of second quantization, the electron wave. The electron wave thus plays a dual and special role, being included both in Cl_3 and $\mathrm{End}(Cl_3)$ by (2.2) and (2.3). So we may say that the electron is both an example of a fermion and the quintessential fermion. The wave $i\phi_n = \phi^8$ is the wave of the neutrino, and also the wave of Lochak's magnetic monopole when it has both a left part and a right part. We previously placed the waves of antiparticles on the second row of each matrix in (2.3) to (2.6) [46, 47]. With the charge conjugation studied in 1.4.1 the second row is determined by the first row of the matrix, and we can use *ad libitum* the complete matrix element of $Cl_{3,1}$ or the first row allowing us to work with $Cl_3 \times Cl_3$. We shall thus employ the convenience of these algebras previously used in [47].

We also saw this essential property of the electron wave: the wave is double, with a right spinor and a left spinor (see (1.60)). The origin of this dualism is the existence of two nonequivalent homomorphisms from the $SL(2,\mathbb{C})$ group into the proper Lorentz group [95]. We also know that not only electrons exist: The β radioactivity that emits electrons also emits another particle nowadays called the electron antineutrino. The electron neutrino and antineutrino induce the existence of another pair of spinors, a left one and a right one. In a theory that unifies all interactions, and since gravitation and the geometry of space-time are strongly linked, the origin of this quartet of spinors which together constitute the lepton wave is necessarily geometric. And the whole of electromagnetism, including the electron wave, is form-invariant under the greater geometric group Cl_3^*. This group is isomorphic to the $GL(2,\mathbb{C})$ group which includes $SL(2,\mathbb{C})$ as a subgroup. The $GL(n,\mathbb{C})$ groups are well known to be the simplest Lie groups, their Lie algebra being the matrix algebra $M_n(\mathbb{C})$. Also well known are the four kinds (not only two) of nonequivalent homomorphisms. Our hypothesis is: these four kinds of homomorphisms are the origin of the existence of the four kinds of spinors forming the lepton wave. In the next chapter we will extend this hypothesis to the quarks.

The Standard Model first considered a neutrino reduced to its left wave only, and without proper mass. Modern experiments on neutrinos show that they must have a proper mass, which is very small indeed yet nonetheless still nonzero, and thus a right neutrino wave must also exist. The Standard Model actually has no objection against the existence of this right wave; it is simply considered useless. Yet the fact that the neutrino travels in space with the speed of light, or in any case with a velocity extraordinarily near light speed, justifies a null proper mass. With the Dirac wave equation for the neutrino, there is thus a problem, which is derived from the mass connection between left and right waves. We will see in this chapter how the improved equation solves all these difficulties.

The starting point of this work was Lochak's theory of the leptonic magnetic monopole [84]-[91], where the wave of the monopole is also a function of space-time with value in Cl_3. Whether for the electron–positron pair or for the electron–neutrino pair or for the electron–monopole pair we obtain in each case four waves with a spinor value, with two left spinors and two right spinors. Moreover, as a particular case, we must again end up with only the electron or only the left electron neutrino. We saw in the first chapter that the charge conjugation is simply the parity–time transformation; hence we will continue to use this in the case of the lepton wave. Thus the two pairs of waves may account for both the electron and the complete neutrino, or for the electron and the magnetic monopole. **This leads us to think that these two objects, the neutrino with proper mass and the magnetic monopole, are the same thing**.

An extension of the Dirac equation for the electroweak interactions [107] was studied by Hestenes [77] and by Boudet [5] [6] in the framework of the $Cl_{1,3}$ algebra. The extension of the gauge invariance of the improved Dirac equation necessarily leads to the gauge invariance under $U(1) \times SU(2)$, also obtained in [12]. This comes from the existence of four independent generators with square -1 in the Cl_3 algebra: $i = \sigma_1\sigma_2\sigma_3$, $i\sigma_1$, $i\sigma_2$ and $i\sigma_3$. They are also the generators of the Lie algebra of $U(1) \times SU(2)$. Since the form invariance is always governed by the Cl_3^* group, and since $Cl_3 \times Cl_3$ is a Cl_3 left modulus, we may express everything in Cl_3: all complex 4×4 matrices of the Dirac theory may be calculated by blocks made of 2×2 matrices.

Under the similitude D generated by any dilator M in Cl_3^* we recall that we have: $\mathrm{x} \mapsto \mathrm{x}' = D(\mathrm{x}) = M\mathrm{x}M^\dagger$, $\nabla = \overline{M}\nabla'\widehat{M}$ and $\det(M) = re^{i\theta}$, and also

$$R^1 \mapsto R'^1 = MR^1;\ \widehat{L}^1 \mapsto \widehat{L}'^1 = \widehat{M}\widehat{L}^1.$$

The R^8 and L^8 waves use the two other homomorphisms with a reversion:

$$\begin{aligned} R^8 &\mapsto R'^{\,8} = R^8\widetilde{M};\ \widetilde{R}'^{\,8} = M\widetilde{R}^8, \\ \widehat{L}^8 &\mapsto \widehat{L}'^{\,8} = \widehat{L}^8\overline{M};\ \overline{L}'^{\,8} = \widehat{M}\overline{L}^8, \end{aligned} \tag{2.8}$$

This gives another reason for the existence of two kinds of leptons in the first generation: the electron and the neutrino. We will see in the next chapter how it also justifies the existence of two quarks in the first generation, the u and the d quarks. The duality between a charged lepton and another lepton without electric charge is reproduced for each one of the three generations, through the simple generalization of Section 1.6. For the four spinors we use the following expressions: [2]:

$$\begin{aligned} &\xi^n = \begin{pmatrix} \xi_1^n \\ \xi_2^n \end{pmatrix};\ \eta^n = \begin{pmatrix} \eta_1^n \\ \eta_2^n \end{pmatrix};\ \widehat{\eta}^n = \begin{pmatrix} -\overline{\eta}_2^n \\ \overline{\eta}_1^n \end{pmatrix};\ \widehat{\xi}^n = \begin{pmatrix} -\overline{\xi}_2^n \\ \overline{\xi}_1^n \end{pmatrix}, \\ &\phi^1 = R^1 + L^1 = \sqrt{2}\begin{pmatrix} \xi^1 & \widehat{\eta}^1 \end{pmatrix};\ R^1 = \sqrt{2}\begin{pmatrix} \xi^1 & 0 \end{pmatrix};\ \widehat{L}^1 = \sqrt{2}\begin{pmatrix} \eta^1 & 0 \end{pmatrix}, \\ &\widetilde{\phi}^8 = \widetilde{R}^8 + \widetilde{L}^8 = \sqrt{2}\begin{pmatrix} \xi^8 & \widehat{\eta}^8 \end{pmatrix};\ \widetilde{R}^8 = \sqrt{2}\begin{pmatrix} \xi^8 & 0 \end{pmatrix};\ \overline{L}^8 = \sqrt{2}\begin{pmatrix} \eta^8 & 0 \end{pmatrix}. \end{aligned} \tag{2.9}$$

We saw that the wave equations of the right and left parts of the electron wave satisfy a first-order equation, with only two extra terms: a gauge term and a mass term. The gauge term is from the geometric point of view a covariant vector, and the mass term is the product of the reduced mass m by a unitary vector v. This vector is the local reduced velocity of the relativistic fluid. The dinum of the different terms allows us to understand why no other term is possible in a first-order wave equation (see 5.6) thus we can only generalize these equations. We can then suppose a similar wave equation for the four waves:

$$i\nabla\eta^1 = l^1\eta^1, \tag{2.10}$$

$$i\widehat{\nabla}\xi^1 = \widehat{r}^1\xi^1, \tag{2.11}$$

$$i\nabla\eta^8 = l^8\eta^8, \tag{2.12}$$

$$i\widehat{\nabla}\xi^8 = \widehat{r}^8\xi^8. \tag{2.13}$$

In this chapter, the M in Cl_3^* which allows us to get the form invariance of the equations is constant. We have $\nabla = \widetilde{\nabla}$ and the four differential operators are reduced to ∇ and $\widehat{\nabla}$. The l^n and r^n are four covariant space-time vectors, and we will see their connection with the potentials of the electroweak gauge group, as well as with the reduced proper mass generalizing the proper mass of the electron.

2. Here we use the usual notation for the complex conjugate.

Let us first see how, when ϕ^8 is zero, the two equations (2.10) (2.11) may be the wave equations of the electron. Since η^1 is, when multiplied by $\sqrt{2}$, the left column of $\widehat{\phi}^1$ while $\widehat{\xi}^1$ is the right column, these equations when multiplied by $\sqrt{2}$ and using the P conjugation on (2.11) read as follows:

$$
\begin{aligned}
i\nabla\widehat{\phi}^1\frac{1+\sigma_3}{2} &= l^1\widehat{\phi}^1\frac{1+\sigma_3}{2},\\
-i\nabla\widehat{\phi}^1\frac{1-\sigma_3}{2} &= r^1\widehat{\phi}^1\frac{1-\sigma_3}{2}.
\end{aligned}
\tag{2.14}
$$

Adding these equations we get

$$
i\nabla\widehat{\phi}^1\sigma_3 = \frac{l^1+r^1}{2}\widehat{\phi}^1 + \frac{l^1-r^1}{2}\widehat{\phi}^1\sigma_3. \tag{2.15}
$$

If we have set

$$
l^1 := q\mathrm{A} + \mathbf{l}\mathrm{v};\ r^1 := q\mathrm{A} + \mathbf{r}\mathrm{v} \tag{2.16}
$$

we have obtained the improved wave equation of the electron [3] :

$$
\nabla\widehat{\phi}^1\sigma_{12} = q\mathrm{A}\widehat{\phi}^1 + \mathrm{v}\widehat{\phi}^1\mathbf{m}, \tag{2.17}
$$

only if the vector v is equal to the J/ρ of Chapter 1, which we will see in the next section.

2.1.1 New tensor densities

In the case of a single electron we used four currents $\mathrm{D}_\mu = \phi\sigma_\mu\phi^\dagger$, particularly the $\mathrm{J} = \mathrm{D}_0 = \phi\phi^\dagger$. This current is the sum of the chiral currents $\mathrm{D}^1_R = R^1\widetilde{R}^1$ and $\mathrm{D}^1_L = L^1\widetilde{L}^1$. Moreover these currents are now similar to two other chiral currents:

$$
\mathrm{D}^8_R := \widetilde{R}^8R^8;\ \mathrm{D}^8_L := \widetilde{L}^8L^8. \tag{2.18}
$$

And these currents, like those of the electron, have a null scalar square:

$$
\begin{aligned}
\mathrm{D}^8_R\cdot\mathrm{D}^8_R &= \mathrm{D}^8_R\widehat{\mathrm{D}}^8_R = \widetilde{R}^8(R^8\overline{R}^8)\widehat{R}^8 = 0,\\
\mathrm{D}^8_L\cdot\mathrm{D}^8_L &= \mathrm{D}^8_L\widehat{\mathrm{D}}^8_L = \widetilde{L}^8(L^8\overline{L}^8)\widehat{L}^8 = 0,
\end{aligned}
\tag{2.19}
$$

because the bracketed quantities are both zero. The natural generalization of the probability current of the electron is the lepton current J_l such that:

$$
J_l = \mathrm{D}^1_R + \mathrm{D}^1_L + \mathrm{D}^8_R + \mathrm{D}^8_L. \tag{2.20}
$$

This current indeed satisfies:

$$
J^0_l = |\xi^1_1|^2 + |\xi^1_2|^2 + |\xi^8_1|^2 + |\xi^8_2|^2 + |\eta^1_1|^2 + |\eta^1_2|^2 + |\eta^8_1|^2 + |\eta^8_2|^2. \tag{2.21}
$$

3. It is well known that the momentum of the electron is the sum of a matter momentum and an electromagnetic momentum. This duality is again obtained here with the densities.

This probability density is the generalization of the density of the electron that we studied in Chapter 1. The time component J_l^0 is now one of $17 \times 16/2 = 136$ tensor densities that we may define without derivatives from our four spinors with four real components each (16, thus $(16+1)16$ densities). **We are now far ahead of the mere 16 tensor densities coming from the $M_4(\mathbb{C})$ algebra generated by the Dirac matrices**, yet presented in most course books as the only possible tensor densities! With (2.19) we have [4]:

$$\begin{aligned}
J_l \cdot J_l = \widehat{J}_l J_l &= (\widehat{\mathrm{D}}^1_R + \widehat{\mathrm{D}}^1_L + \widehat{\mathrm{D}}^8_R + \widehat{\mathrm{D}}^8_L)(\mathrm{D}^1_R + \mathrm{D}^1_L + \mathrm{D}^8_R + \mathrm{D}^8_L) \\
&= \widehat{\mathrm{D}}^1_R \mathrm{D}^1_R + \widehat{\mathrm{D}}^1_L \mathrm{D}^1_L + \widehat{\mathrm{D}}^8_R \mathrm{D}^8_R + \widehat{\mathrm{D}}^8_L \mathrm{D}^8_L \\
&+ \widehat{\mathrm{D}}^1_R \mathrm{D}^1_L + \widehat{\mathrm{D}}^1_L \mathrm{D}^1_R + \widehat{\mathrm{D}}^1_R \mathrm{D}^8_R + \widehat{\mathrm{D}}^8_R \mathrm{D}^1_R + \widehat{\mathrm{D}}^1_R \mathrm{D}^8_L + \widehat{\mathrm{D}}^8_L \mathrm{D}^1_R \\
&+ \widehat{\mathrm{D}}^1_L \mathrm{D}^8_R + \widehat{\mathrm{D}}^8_R \mathrm{D}^1_L + \widehat{\mathrm{D}}^1_L \mathrm{D}^8_L + \widehat{\mathrm{D}}^8_L \mathrm{D}^1_L + \widehat{\mathrm{D}}^8_R \mathrm{D}^8_L + \widehat{\mathrm{D}}^8_L \mathrm{D}^8_R \qquad (2.22)\\
&= 2(\mathrm{D}^1_R \cdot \mathrm{D}^1_L + \mathrm{D}^1_R \cdot \mathrm{D}^8_R + \mathrm{D}^1_R \cdot \mathrm{D}^8_L + \mathrm{D}^1_L \cdot \mathrm{D}^8_R + \mathrm{D}^1_L \cdot \mathrm{D}^8_L + \mathrm{D}^8_R \cdot \mathrm{D}^8_L).
\end{aligned}$$

We saw that $2\mathrm{D}^1_R \cdot \mathrm{D}^1_L = a_1 a_1^*$ where $a_1 = \Omega_1 + i\Omega_2$, a term formed from the two relativistic invariants Ω_1 and Ω_2 of the electron wave. With four spinors we form $6 = 4 \times 3/2$ pairs, each giving a term similar to a_1. Thus we now have 12 invariant densities which give the 6 complex terms:

$$\begin{aligned}
a_1 &= 2(\xi^1_1 \overline{\eta}^1_1 + \xi^1_2 \overline{\eta}^1_2) = R^1 \overline{L}^1 + L^1 \overline{R}^1, \\
a_2 &= 2(\eta^8_1 \eta^1_2 - \eta^8_2 \eta^1_1) = \widehat{L}^1 \sigma_1 L^8 - \overline{L}^8 \sigma_1 \widetilde{L}^1, \\
a_3 &= 2(\xi^1_1 \overline{\eta}^8_1 + \xi^1_2 \overline{\eta}^8_2) = R^1 \widehat{L}^8 + \widetilde{L}^8 \overline{R}^1, \\
a_4 &= 2(\xi^8_1 \overline{\eta}^1_1 + \xi^8_2 \overline{\eta}^1_2) = \widetilde{R}^8 \overline{L}^1 + L^1 \widehat{R}^8, \qquad (2.23)\\
a_5 &= 2(\xi^1_1 \xi^8_2 - \xi^1_2 \xi^8_1) = \widetilde{R}^8 \sigma_1 \overline{R}^1 - R^1 \sigma_1 \widehat{R}^8, \\
a_6 &= 2(\xi^8_1 \overline{\eta}^8_1 + \xi^8_2 \overline{\eta}^8_2) = \widetilde{R}^8 \widehat{L}^8 + \widetilde{L}^8 \widehat{R}^8.
\end{aligned}$$

In a similitude D generated by any dilator M in Cl_3^* we have for $j = 1, 2, \dots, 6$:

$$a'_j = M a_j \overline{M} = M \overline{M} a_j = r e^{i\theta} a_j, \qquad (2.24)$$

$$a'_j a'^*_j = r e^{i\theta} a_j r e^{-i\theta} a_j^* = r^2 a_j a_j^*. \qquad (2.25)$$

We may then generalize the invariant ρ of the wave of the electron into ρ_l such that:

$$\rho_l^2 = \sum_{j=1}^{6} a_j a_j^*, \qquad (2.26)$$

4. It is thus surprising and disturbing that this significant mistake concerning the number of tensor densities in the Dirac theory, pointed out since nearly thirty years ago [16] by one of the present authors, is not yet corrected. This means that the control of errors by the same "community" which propagated these errors, is imperfect, inefficient [38], and much too conservative.

which satisfies as ρ previously:

$$\rho_l'^2 = r^2\rho_l^2;\ \rho_l' = r\rho_l;\ m'\rho_l' = m\rho_l. \tag{2.27}$$

Both a_j and ρ_l thus have a dinum of 1 and m has a dinum of -1 (see 1.7). In the domain of weak interactions the neutrino appears without a mass term. It is necessary to add a mass term when physicists try to understand the behaviour of neutrinos changing generation. And we may remark that $m\rho_l$ is null when ρ_l is null, and this happens in the case where the neutrino only has a left wave. The currents thus satisfy $J_l = \mathrm{D}_L^8$ and $\mathrm{D}_L^8 \cdot \mathrm{D}_L^8 = 0$, and hence $m\rho_l = 0$: **the single left neutrino appears without mass**. The mass term again appears as soon as R^8 is not null and as soon as the wave of the electron is not null. The behavior of this mass which appears and disappears *ad libitum* is not so mysterious if we consider that we are able to see a neutrino only when it interacts with a charged lepton or a quark. For the electron we have $(J)^2 = \rho^2 = a_1a_1^* = 2\mathrm{D}_R^1 \cdot \mathrm{D}_L^1$. Similarly we obtain:

$$a_1a_1^* = 2\mathrm{D}_R^1 \cdot \mathrm{D}_L^1, \tag{2.28}$$

$$a_5a_5^* = 2\mathrm{D}_R^1 \cdot \mathrm{D}_R^8, \tag{2.29}$$

because we have

$$\begin{aligned}
2\mathrm{D}_R^1 \cdot \mathrm{D}_R^8 &= \mathrm{D}_R^1\widehat{\mathrm{D}}_R^8 + \mathrm{D}_R^8\widehat{\mathrm{D}}_R^1 = R^1\widetilde{R}^1\overline{R}^8\widehat{R}^8 + \widetilde{R}^8R^8\widehat{R}^1\overline{R}^1,\\
\widetilde{R}^1\overline{R}^8 &= \begin{pmatrix}0 & -a_5^*\\ 0 & 0\end{pmatrix};\ R^8\widehat{R}^1 = \begin{pmatrix}0 & a_5^*\\ 0 & 0\end{pmatrix},
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
2\mathrm{D}_R^1 \cdot \mathrm{D}_R^8 &= R^1\begin{pmatrix}0 & -a_5^*\\ 0 & 0\end{pmatrix}\widehat{R}^8 + \widetilde{R}^8\begin{pmatrix}0 & a_5^*\\ 0 & 0\end{pmatrix}\overline{R}^1\\
&= (\widetilde{R}^8\sigma_1\overline{R}^1 - R^1\sigma_1\widehat{R}^8)a_5^* = a_5a_5^*.
\end{aligned} \tag{2.31}$$

We can likewise establish:

$$\begin{aligned}
2\mathrm{D}_R^1 \cdot \mathrm{D}_L^8 &= a_3a_3^*;\ 2\mathrm{D}_L^1 \cdot \mathrm{D}_R^8 = a_4a_4^*,\\
2\mathrm{D}_L^1 \cdot \mathrm{D}_L^8 &= a_2a_2^*;\ 2\mathrm{D}_R^8 \cdot \mathrm{D}_L^8 = a_6a_6^*.
\end{aligned} \tag{2.32}$$

We then derive from these equations and from (2.23), (2.24) and (2.26) that we have:

$$J_l \cdot J_l = \rho_l^2;\ ||J_l|| = \rho_l. \tag{2.33}$$

As for the single electron we may define a unitary vector v as

$$\mathrm{v} = \frac{J_l}{\rho_l} = \mathrm{v}^\mu\sigma_\mu;\ \mathrm{v}\widehat{\mathrm{v}} = \widehat{\mathrm{v}}\mathrm{v} = 1;\ \widehat{\mathrm{v}} = \mathrm{v}^{-1}. \tag{2.34}$$

We retain the same notation v as for the case of the electron because this vector is exactly that of Chapter 1 when the wave of the neutrino is null.

The natural generalization of the Lagrangian density of the electron is thus able to contain the same mass terms, and the wave equation is able to contain the same v in mass terms.

We must never forget that the previous tensors are only a small part of the many tensor densities that we are able to construct from the spinor wave, left and right. The differentiation of these tensors yields new ones, which in turn give others by deriving again, *ad infinitum* – unhappily [16]. We will also use some of these other tensor densities, for instance the energy–momentum tensors.

2.1.2 The electroweak gauge invariance

We begin with the case of the electron following [67]. We modify nothing to the wave of the electron which we denoted as ψ^1 in the usual Dirac formalism and ϕ^1 in Cl_3. The wave of the electron neutrino is denoted as ψ^8 in the Dirac formalism and $\widetilde{\phi}^8$ in Cl_3. The wave of the positron is denoted as ψ_p in the Dirac formalism and the wave of the electron antineutrino is denoted as ψ_a. The link between the particle and antiparticle wave will remain the previous link seen in 1.4.1. Then we start with the particle waves. The right spinors are ξ^n and the left spinors are η^n:

$$\psi^1 = \begin{pmatrix} \xi^1 \\ \eta^1 \end{pmatrix} \ ; \ \psi^8 = \begin{pmatrix} \xi^8 \\ \eta^8 \end{pmatrix}. \tag{2.35}$$

We again use the notation of (2.9) which gives us

$$\phi^1 = \sqrt{2}\begin{pmatrix} \xi^1 & \widehat{\eta}^1 \end{pmatrix} ; \ \widehat{\phi}^1 = \sqrt{2}\begin{pmatrix} \eta^1 & \widehat{\xi}^1 \end{pmatrix}, \tag{2.36}$$

$$\widetilde{\phi}^8 = \sqrt{2}\begin{pmatrix} \xi^8 & \widehat{\eta}^8 \end{pmatrix} ; \ \overline{\phi}^8 = \sqrt{2}\begin{pmatrix} \eta^8 & \widehat{\xi}^8 \end{pmatrix}. \tag{2.37}$$

With the link that the Standard Model makes between the particle and the antiparticle wave, using $Cl_{3,1}$ and the shortened notation of $Cl_3 \times Cl_3$, we have:

$$\Psi_l = \begin{pmatrix} \phi^1 & \widetilde{\phi}^8 \\ -\overline{\phi}^8 & \widehat{\phi}^1 \end{pmatrix} = \begin{pmatrix} \phi^1 & \widetilde{\phi}^8 \end{pmatrix}. \tag{2.38}$$

The Weinberg–Salam model uses ξ^1, η^1, η^8 and supposes ξ^8 to be null. We will use the complete wave for Lochak's magnetic monopole while the neutrino itself will have no right wave. So we consider the magnetic monopole as a complete neutrino with a left and right wave and we consider the neutrino of the SM as a monopole in which the right wave is absent. For the separation of ξ^1, η^1 and η^8 the Weinberg–Salam model uses the $\frac{1}{2}(1 \pm \gamma_5)$ projectors that can be presented as follows, with our choice (1.4) of Dirac matrices:

$$\frac{1}{2}(1-\gamma_5)\psi = \psi_L = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \tag{2.39}$$

$$\frac{1}{2}(1+\gamma_5)\psi = \psi_R = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix}. \tag{2.40}$$

Thus for the particles the left waves are η waves and the right waves are ξ waves. This is invariant under Cl_3^* and therefore relativistically invariant, since under a similitude D generated by M such that $D : \mathrm{x} \mapsto \mathrm{x}' = M\mathrm{x}M^\dagger$, we have (1.60): $\xi' = M\xi$, $\eta' = \widehat{M}\eta$. So we use

$$R^1 = \sqrt{2}\begin{pmatrix} \xi^1 & 0 \end{pmatrix} = \sqrt{2}\begin{pmatrix} \xi^1 & \widehat{\eta}^1 \end{pmatrix} \frac{1}{2}(1+\sigma_3),$$
$$L^1 = \sqrt{2}\begin{pmatrix} 0 & \widehat{\eta}^1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} \xi^1 & \widehat{\eta}^1 \end{pmatrix} \frac{1}{2}(1-\sigma_3). \tag{2.41}$$

And we get similar formulas for $\widetilde{R}^8$ and $\widetilde{L}^8$. We now define two projectors $P_\pm$ and four operators P_0, P_1, P_2, P_3 acting as follows on $\Psi \in Cl_3 \times Cl_3$:

$$P_\pm(\Psi) = \frac{1}{2}(\Psi \pm \mathbf{i}\Psi\gamma_{21}) \; ; \; \mathbf{i} = (i \;\; 0); \; \gamma_{21} = (\sigma_{12} \;\; 0), \tag{2.42}$$

Thus we get:

$$P_+(\Psi_l) = \begin{pmatrix} L^1 & \widetilde{L}^8 \end{pmatrix}; \; P_-(\Psi_l) = \begin{pmatrix} R^1 & \widetilde{R}^8 \end{pmatrix}. \tag{2.43}$$

So P_+ is the projector on the left part of the wave and P_- is the projector on the right part of the wave. We let:

$$P_0(\Psi) = \Psi\gamma_{21} + (1-p)P_-(\Psi)\mathbf{i} + p\mathbf{i}P_-(\Psi), \tag{2.44}$$
$$P_1(\Psi) = \mathbf{i}P_+(\Psi)\gamma_3\gamma_5, \tag{2.45}$$
$$P_2(\Psi) = \mathbf{i}P_+(\Psi)(-i\gamma_3), \tag{2.46}$$
$$P_3(\Psi) = P_+(\Psi)(-\mathbf{i}). \tag{2.47}$$

We introduced here a number p which is linked to the charge of the magnetic monopole and which acts only on the right wave of the neutrino, which is unknown in the Standard Model. Noting $P_\mu P_\nu(\Psi) = P_\mu[P_\nu(\Psi)]$, they satisfy:

$$P_1P_2 = P_3 = -P_2P_1; \; P_2P_3 = P_1 = -P_3P_2; \; P_3P_1 = P_2 = -P_1P_3, \tag{2.48}$$
$$P_1^2 = P_2^2 = P_3^2 = -P_+; \; P_0P_j = P_jP_0 = -\mathbf{i}P_j \;, \quad j = 1, \; 2, \; 3.$$

The Weinberg–Salam model replaces the ∂_μ derivatives with the covariant derivatives:

$$\mathrm{D}_\mu = \partial_\mu - ig_1\frac{Y}{2}B_\mu - ig_2T_jW_\mu^j, \tag{2.49}$$

with $T_j = \tau_j/2$ for a doublet of left particles and $T_j = 0$ for a singlet of right [5] particles. Y is the weak hypercharge, with $Y_L = -1$, $Y_R = -2$ for the electron. For transposing this to $Cl_3 \times Cl_3$ we let:

$$\mathrm{D} = \sigma^\mu \mathrm{D}_\mu;\ B = \sigma^\mu B_\mu;\ W^j = \sigma^\mu W^j_\mu,\ j = 1,2,3, \tag{2.50}$$

$$\mathbf{D} = \gamma^\mu \mathrm{D}_\mu;\ \mathbf{B} = \gamma^\mu B_\mu;\ \mathbf{W}^j = \gamma^\mu W^j_\mu,\ j = 1,2,3;\ \boldsymbol{\partial} = \gamma^\mu \partial_\mu. \tag{2.51}$$

We now replace (2.49) by:

$$\mathbf{D} = \boldsymbol{\partial} + \frac{g_1}{2}\mathbf{B}P_0 + \frac{g_2}{2}(\mathbf{W}^1 P_1 + \mathbf{W}^2 P_2 + \mathbf{W}^3 P_3). \tag{2.52}$$

First we have

$$\boldsymbol{\partial}\Psi_l = \begin{pmatrix} -\nabla\overline{\phi}^8 & \nabla\widehat{\phi}^1 \end{pmatrix}, \tag{2.53}$$

$$\mathbf{D}\Psi_l = \begin{pmatrix} -\mathrm{D}\overline{\phi}^8 & \mathrm{D}\widehat{\phi}^1 \end{pmatrix}. \tag{2.54}$$

And we get:

$$P_0(\Psi_l) = i\begin{pmatrix} 2R^1 - L^1 & 2p\widetilde{R}^8 - \widetilde{L}^8 \end{pmatrix}. \tag{2.55}$$

From the form of these P_μ we may see that the Weinberg–Salam model of weak interactions using only R^1, L^1 and L^8 does not depend on the value of p that may be any number. We will later see how this value is linked to the charge of the magnetic monopole. We next obtain:

$$\mathbf{B}P_0(\Psi_l) = \begin{pmatrix} iB(2p\overline{R}^8 - \overline{L}^8) & iB(-2\widehat{R}^1 + \widehat{L}^1) \end{pmatrix}. \tag{2.56}$$

Next we have

$$P_1(\Psi_l) = \begin{pmatrix} -i\widetilde{L}^8 & iL^1 \\ i\widehat{L}^1 & -i\overline{L}^8 \end{pmatrix};\ \mathbf{W}^1 P_1(\Psi_l) = \begin{pmatrix} iW^1\widehat{L}^1 & -iW^1\overline{L}^8 \end{pmatrix}, \tag{2.57}$$

$$P_2(\Psi_l) = \begin{pmatrix} \widetilde{L}^8 & -L^1 \\ -\widehat{L}^1 & -\overline{L}^8 \end{pmatrix};\ \mathbf{W}^2 P_2(\Psi_l) = \begin{pmatrix} W^2\widehat{L}^1 & W^2\overline{L}^8 \end{pmatrix}. \tag{2.58}$$

We get for $j = 3$:

$$P_3(\Psi_l) = \begin{pmatrix} -iL^1 & i\widetilde{L}^8 \\ i\overline{L}^8 & i\widehat{L}^1 \end{pmatrix};\ \mathbf{W}^3 P_3(\Psi_l) = \begin{pmatrix} iW^3\overline{L}^8 & iW^3\widehat{L}^1 \end{pmatrix}. \tag{2.59}$$

Therefore (2.52) is equivalent to the system:

$$\begin{aligned} D\overline{\phi}^8 &= \nabla\overline{\phi}^8 - i\frac{g_1}{2}B(2p\overline{R}^8 - \overline{L}^8) - i\frac{g_2}{2}[(W^1 - iW^2)\widehat{L}^1 + W^3\overline{L}^8], \\ D\widehat{\phi}^1 &= \nabla\widehat{\phi}^1 - i\frac{g_1}{2}B(2\widehat{R}^1 - \widehat{L}^1) - i\frac{g_2}{2}[(W^1 + iW^2)\overline{L}^8 - W^3\widehat{L}^1. \end{aligned} \tag{2.60}$$

5. This preference for the left waves is presupposed and not explained in the Weinberg–Salam model. We will explain the origin of this preference in 3.8 .

Since ξ^1 is the left column of R^1, and ξ^8 is the left column of $\widetilde{R}^8$, while η^1 is the left column of $\widehat{L}^1$, and η^8 is the left column of $\overline{L}^8$ (not forgetting a $\sqrt{2}$ factor), this system gives for the particles (electrons and neutrinos) and using the main automorphism $P : M \mapsto \widehat{M}$ for the right waves:

$$\begin{aligned}
\widehat{D}R^1 &= \widehat{\nabla}R^1 + ig_1\widehat{B}R^1,\\
D\widehat{L}^1 &= \nabla\widehat{L}^1 + i\frac{g_1}{2}B\widehat{L}^1 - \frac{ig_2}{2}[(W^1 + iW^2)\overline{L}^8 - W^3\widehat{L}^1],\\
D\overline{L}^8 &= \nabla\overline{L}^8 + \frac{ig_1}{2}B\overline{L}^8 - \frac{ig_2}{2}[(W^1 - iW^2)\widehat{L}^1 + W^3\overline{L}^8];\\
\widehat{D}\widetilde{R}^8 &= \widehat{\nabla}\widetilde{R}^8 + ipg_1\widehat{B}\widetilde{R}^8.
\end{aligned} \tag{2.61}$$

For the waves of the positron and the antineutrino we similarly obtain

$$\begin{aligned}
D\widehat{L}^{\overline{1}} &= \nabla\widehat{L}^{\overline{1}} - ig_1B\widehat{L}^{\overline{1}},\\
\widehat{D}R^{\overline{1}} &= \widehat{\nabla}R^{\overline{1}} - \frac{ig_1}{2}\widehat{B}R^{\overline{1}} - \frac{ig_2}{2}[(\widehat{W}^1 - i\widehat{W}^2)\widetilde{R}^{\overline{8}} + \widehat{W}^3R^{\overline{1}}],\\
\widehat{D}\widetilde{R}^{\overline{8}} &= \widehat{\nabla}\widetilde{R}^{\overline{8}} - i\frac{g_1}{2}\widehat{B}\widetilde{R}^{\overline{8}} - i\frac{g_2}{2}[(\widehat{W}^1 + i\widehat{W}^2)R^{\overline{1}} - \widehat{W}^3\widetilde{R}^{\overline{8}}],\\
D\overline{L}^{\overline{8}} &= \nabla\overline{L}^{\overline{8}} - ig_1pB\overline{L}^{\overline{8}}.
\end{aligned} \tag{2.62}$$

The system (2.61) is equivalent to:

$$D_\mu\xi^1 = \partial_\mu\xi^1 + ig_1B_\mu\xi^1, \tag{2.63}$$
$$D_\mu\eta^1 = \partial_\mu\eta^1 + i\frac{g_1}{2}B_\mu\eta^1 - i\frac{g_2}{2}[(W^1_\mu + iW^2_\mu)\eta^8 - W^3_\mu\eta^1], \tag{2.64}$$
$$D_\mu\eta^8 = \partial_\mu\eta^8 + i\frac{g_1}{2}B_\mu\eta^8 - i\frac{g_2}{2}[(W^1_\mu - iW^2_\mu)\eta^1 + W^3_\mu\eta^8], \tag{2.65}$$
$$D_\mu\xi^8 = \partial_\mu\xi^8 + ig_1pB_\mu\xi^8,\ \mu = 0,1,2,3, \tag{2.66}$$

for the particle waves (2.62) is likewise equivalent to:

$$D_\mu\xi^{\overline{8}} = \partial_\mu\xi^{\overline{8}} - i\frac{g_1}{2}B_\mu\xi^{\overline{8}} - i\frac{g_2}{2}[(W^1_\mu + iW^2_\mu)\xi^{\overline{1}} - W^3_\mu\xi^{\overline{8}}], \tag{2.67}$$
$$D_\mu\eta^{\overline{8}} = \partial_\mu\eta^{\overline{8}} - ig_1pB_\mu\eta^{\overline{8}}, \tag{2.68}$$
$$D_\mu\eta^{\overline{1}} = \partial_\mu\eta^{\overline{1}} - ig_1B_\mu\eta^{\overline{1}},\ \mu = 0,1,2,3, \tag{2.69}$$
$$D_\mu\xi^{\overline{1}} = \partial_\mu\xi^{\overline{1}} - i\frac{g_1}{2}B_\mu\xi^{\overline{1}} - i\frac{g_2}{2}[(W^1_\mu - iW^2_\mu)\xi^{\overline{8}} + W^3_\mu\xi^{\overline{1}}], \tag{2.70}$$

for the antiparticles waves. For the doublet $\psi_L = \begin{pmatrix}\eta^8\\ \eta^1\end{pmatrix}$ of weak isospin $Y = -1$ the operators in (2.64) and (2.65) give

$$\begin{aligned}
D_\mu\psi_L &= \partial_\mu\psi_L - ig_1\frac{Y}{2}B_\mu\psi_L - i\frac{g_2}{2}W^j_\mu\tau_j\psi_L,\\
\tau_1 &= \gamma_0\ ;\ \tau_2 = \gamma_{123}\ ;\ \tau_3 = \gamma_5.
\end{aligned} \tag{2.71}$$

Only the operator in (2.66) is not accounted for the Weinberg–Salam model because this model does not use the right wave of the electron neutrino. We precisely arrive at this model if this right wave is null. The operator in (2.63) is interpreted as a singlet under $SU(2)$: $\psi_R = \xi^1$ with weak isospin $Y = -2$:

$$D_\mu \psi_R = \partial_\mu \psi_R - ig_1 \frac{Y}{2} B_\mu \psi_R. \tag{2.72}$$

Finally we see here that all aspects of the weak interactions, with a doublet of left waves, a right wave which is a singlet, a right neutrino unable to interact, a charge conjugation exchanging right and left waves, are obtained here from simple hypotheses:

1 - The wave of all components of the lepton sector (electron, positron, electron neutrino and its antineutrino) is the function (2.38) of space and time with values in the Clifford algebra of space-time.

2 - Four operators P_0, P_1, P_2 and P_3 are defined in (2.44) to (2.47).

3 - A covariant derivative is defined in (2.52).

For the antiparticles, in the case where the wave of the magnetic monopole is reduced to the neutrino wave, we have a singulet of left wave and a doublet of right waves. By letting:

$$\psi_{\overline{L}} = \eta^{\overline{1}};\ \psi_{\overline{R}} = \begin{pmatrix} \xi^{\overline{8}} \\ \xi^{\overline{1}} \end{pmatrix};\ \overline{\tau}_1 = -\gamma_0;\ \overline{\tau}_2 = \gamma_{123};\ \overline{\tau}_3 = \gamma_5, \tag{2.73}$$

we get

$$D_\mu \psi_{\overline{L}} = \partial_\mu \psi_{\overline{L}} - ig_1 \frac{Y}{2} B_\mu \psi_{\overline{L}}, \tag{2.74}$$

with a weak isospin $Y = 2$, in accordance with the usual rule changing charge signs. For the doublet of right waves we get

$$D_\mu \psi_{\overline{R}} = \partial_\mu \psi_{\overline{R}} - ig_1 \frac{Y}{2} B_\mu \psi_{\overline{R}} + i\frac{g_2}{2} W^j_\mu \overline{\tau}_j \psi_{\overline{R}}. \tag{2.75}$$

The rule of the change of signs for all charges is equivalent to the change of sign for $g_1 Y$ and g_2. But these rules are not sufficient; another change of sign concerns $\overline{\tau}_1 = -\tau_1$. This calls for two remarks: First, the $SU(2)$ gauge group thought of by quantum theory as an "internal symmetry" is indeed a geometrical invariance group. This is completely forgotten when in this exposition we pass from (2.62), where equations contain space-time vectors, to (2.63)–(2.70) where equations only have components of tensors: these tensors are no longer constrained by invariance under Cl_3^* but only by invariance under the Lorentz group. Indeed the two points of view are not equivalent: the point of view of the Weinberg–Salam model is less constrained, less efficient than the Clifford algebra. The result is that our equations are Cl_3^* invariant, but not the Weinberg–Salam model that is in fact disconnected from the relativistic invariance of the fermion field: The relativistic invariance works classically for the gauge fields (where the Lorentz transformation R

defines a 4×4 real matrix R^{μ}_{ν} which changes x^{μ} into $x'^{\mu} = R^{\mu}_{\nu}x^{\nu}$, and where the electromagnetic field $F_{\mu\nu}$ changes into $F'_{\rho\sigma} = R^{\mu}_{\rho}R^{\nu}_{\sigma}F_{\mu\nu}$), and in a quantum-mechanical manner for the spinor waves (where the dilator M induces a similitude R which changes F into $F' = MFM^{-1}$, which is not at all the same transformation). The new connection that we establish here between the fermionic wave of the electron and its neutrino and the tensors that they allow us to construct, connects even more the fermion part and the boson part of the Standard Model. This allows us to arrive at a unified synthesis between the different parts of relativistic physics, which was impossible with the old tensor-based theory.

With charge conjugation simply acting like PT symmetry, space changes orientation. Thus the three τ_j rotate inversely from the $\overline{\tau}_j$, as is shown by the sign change of τ_1.

2.2 Retaining mass terms

The first improvement that Cl_3 brings to quantum mechanics in the Dirac matrix style is the possibility of also using the right R^8 spinor that we associate with the magnetic monopole. A second and major improvement: we no longer need to suppress mass terms in wave equations. This suppression was necessary when using the usual Dirac equation, because mass terms link the left and the right wave, while ξ and η change in very different ways under the gauge transformations of the electroweak group [39] [40] [41] [44].

This suppression was also an acceptable lesser evil, from the experimental point of view, because proper masses [6] of the electron and, still more, proper masses of the electron neutrino are very small in comparison with the mass-energy of the W and Z^0 bosons. Nevertheless this suppression is necessarily an approximation since the electron *has* mass-energy, and since the wave equation of the neutrino probably also has mass terms. Since it was impossible to account for both proper mass and electroweak gauge, a mechanism of spontaneous symmetry breaking was constructed. The Higgs boson (which was thought of as able to reintroduce masses into wave equations) was finally observed at very high energy ($\approx$ 126 GeV). However, this still does not transform the electroweak gauge into a theory compatible with mass and gravitation. In reality each proper mass is replaced by a coupling coefficient with the Higgs boson. So the Higgs boson, even though it exists, does not explain much. And the existence of such a scalar field with high mass was suspected as early as de Broglie's theory of the photon [55, 56].

What we now know how to do is very different and much more innovative since we are able to restore the compatibility of the covariant electroweak derivation that we just studied with equations (2.10) to (2.13). We are able

6. The improved wave equation tolerates two different mass terms for the electron (see Chapter 1).

to restore this because the improved equation that we obtained as a wave equation may be recast into a form that seems uncrossed and acting on only one chiral spinor. And this is easily generalized. Maintaining mass terms in the wave equation will allow us in Chapter 4 to directly put together gravitation and other forces in the wave equations. It is thus an important improvement towards the unification of all interactions. We are even able to conserve the form of the mass term $\mathrm{v}\widehat{\phi}^1\mathbf{m}$ of the improved wave equation:

$$\begin{aligned}&0=-iD\widehat{L}^1+\mathbf{l}\mathrm{v}\widehat{L}^1;\ 0=-i\widehat{D}R^1+\mathbf{r}\widehat{\mathrm{v}}R^1,\\&0=-i\widetilde{D}\overline{L}^8+m_l\mathrm{v}\overline{L}^8;\ 0=-i\overline{D}\widetilde{R}^8+m_r\widehat{\mathrm{v}}\widetilde{R}^8.\end{aligned}\tag{2.76}$$

We do not suppose that the $\mathbf{l}$, $\mathbf{r}$, m_l and m_r coefficients are all equal. The unitary vector v remains defined from the four spinor waves by (2.34). We simplify the following study by considering only the wave of the electron and of the neutrino–monopole, as a beginning. We will derive the properties of the positron and antineutrino–monopole by changing the sign of the differential terms of the wave equation, and exchanging η and ξ terms. With the form obtained in (2.61) for the derivation with gauge terms, the wave equations (2.76) become:

$$0=\widehat{\nabla}R^1+ig_1\widehat{B}R^1+i\mathbf{r}\widehat{\mathrm{v}}R^1,\tag{2.77}$$
$$0=\nabla\widehat{L}^1+i\frac{g_1}{2}B\widehat{L}^1-i\frac{g_2}{2}[(W^1+iW^2)\overline{L}^8-W^3\widehat{L}^1]+i\mathbf{l}\mathrm{v}\widehat{L}^1,\tag{2.78}$$
$$0=\widetilde{\nabla}\overline{L}^8+i\frac{g_1}{2}B\overline{L}^8-i\frac{g_2}{2}[(W^1-iW^2)\widehat{L}^1+W^3\overline{L}^8]+im_l\mathrm{v}\overline{L}^8,\tag{2.79}$$
$$0=\overline{\nabla}\widetilde{R}^8+ig_1p\widehat{B}\widetilde{R}^8+im_r\widehat{\mathrm{v}}\widetilde{R}^8.\tag{2.80}$$

We may remark that the coefficients of B are the same only for L^1 and L^8. So left waves, turning in the same manner in the chiral gauge, can be mixed in the $SU(2)$ gauge group. Comparing with potentials we may see that these equations are indeed wave equations with two different spinors. We now see how it is possible to use the invariance of each wave equation.

2.3 Extended invariance

With the similitude induced by any dilator M in Cl_3:

$$\begin{aligned}\mathrm{D}'^1_R&=R'^1\widetilde{R}'^1=MR^1\widetilde{MR}^1=MR^1\widetilde{R}^1\widetilde{M}=M\mathrm{D}^1_R\widetilde{M},\\\mathrm{J}'_l&=M\mathrm{J}_l\widetilde{M};\ \rho'=r\rho;\ m=m'r;\ (m=\mathbf{l},\mathbf{r},m_l,m_r),\end{aligned}\tag{2.81}$$

$$\begin{aligned}m'\mathrm{v}'&=\frac{m}{r}\frac{\mathrm{J}'_l}{\rho'}=\frac{m}{r}\frac{M\mathrm{J}_l\widetilde{M}}{r\rho}=m\frac{M}{re^{i\theta}}\mathrm{v}\frac{\widetilde{M}}{re^{-i\theta}}=m\overline{M}^{-1}\mathrm{v}\widehat{M}^{-1},\\\overline{M}m'\mathrm{v}'\widehat{L}'^1&=m\mathrm{v}\widehat{M}^{-1}\widehat{M}\widehat{L}^1=m\mathrm{v}\widehat{L}^1,\end{aligned}\tag{2.82}$$

we simplify the wave equations (2.77)–(2.80) with:

$$\begin{aligned}
&\mathrm{p}_L^1 = \frac{g_1}{2}B + \mathbf{l}\mathrm{v} = \mathrm{b} + \mathbf{l}\mathrm{v};\ \mathrm{w}^j = \frac{g_2}{2}W^j,\ j = 1,2,3,\\
&\mathrm{p}_R^1 = g_1 B + \mathbf{r}\mathrm{v} = 2\mathrm{b} + \mathbf{r}\mathrm{v};\ \mathrm{p}_L^8 = \frac{g_1}{2}B + m_l\mathrm{v} = \mathrm{b} + m_l\mathrm{v},\\
&\mathrm{p}_R^8 = g_1 pB + m_r\mathrm{v} = 2p\mathrm{b} + m_r\mathrm{v}.
\end{aligned} \tag{2.83}$$

To obtain the relativistic invariance of the equation of L^1, for instance, we must have for the gauge potentials the same variance as the differential term. And this term is covariant which means it satisfies $\nabla = \overline{M}\nabla'\widehat{M}$. It is the same with p, b and for the w^j which are also covariant vectors because these vectors incorporate the g_1 and g_2 charges. We have:

$$0 = \widehat{\nabla}R^1 + i\widehat{\mathrm{p}}_R^1 R^1 = \widetilde{M}[\widehat{\nabla}' R'^1 + i\widehat{\mathrm{p}}'^1_R R'^1], \tag{2.84}$$

$$\begin{aligned}
0 &= \nabla\widehat{L}^1 + i\mathrm{p}_L^1\widehat{L}^1 - i[(\mathrm{w}^1 + i\mathrm{w}^2)\overline{L}^8 - \mathrm{w}^3\widehat{L}^1)]\\
&= \overline{M}\Big[\nabla\widehat{L}'^1 + i\mathrm{p}'^1_L\widehat{L}'^1 - i[(\mathrm{w}'^1 + i\mathrm{w}'^2)\overline{L}'^8 - \mathrm{w}'^3\widehat{L}'^1]\Big];
\end{aligned} \tag{2.85}$$

$$\begin{aligned}
0 &= \nabla\overline{L}^8 + i\mathrm{p}_L^8\overline{L}^8 - i[(\mathrm{w}^1 - i\mathrm{w}^2)\widehat{L}^1 + \mathrm{w}^3\overline{L}^8)];\\
&= \overline{M}\Big[\nabla\overline{L}'^8 + i\mathrm{p}'^8_L)\overline{L}'^8 - i[(\mathrm{w}'^1 - i\mathrm{w}'^2)\widehat{L}'^1 + \mathrm{w}'^3\overline{L}^8]\Big];
\end{aligned} \tag{2.86}$$

$$0 = \widehat{\nabla}\widetilde{R}^8 + i\widehat{\mathrm{p}}_R^8\widetilde{R}^8 = \widetilde{M}\big[\widehat{\nabla}'\widetilde{R}'^8 + i\widehat{\mathrm{p}}'^8_R\widetilde{R}'^8\big]. \tag{2.87}$$

This provides the form invariance of the wave equations, as in the case of the lone electron that we studied in Chapter 1.

The gauge transformations are generated by P_0, P_1, P_2 and P_3. This gives to us a group with four parameters a^0, a^1, a^2 and a^3. We recall the definition of the exponential function

$$\exp(a^0P_0) = \sum_{n=0}^{\infty}\frac{(a^0P_0)^n}{n!};\quad \exp(a^jP_j) = \sum_{n=0}^{\infty}\frac{(a^1P_1 + a^2P_2 + a^3P_3)^n}{n!}. \tag{2.88}$$

Since these operators were defined in $Cl_3 \times Cl_3$ and since they are different for right and left waves we will study them with the $Cl_3 \times Cl_3$ form (see B.1.2) of these right and left spinors:

$$\Psi_R^1 = \begin{pmatrix}R^1 & 0\end{pmatrix};\ \Psi_L^1 = \begin{pmatrix}L^1 & 0\end{pmatrix};\ \Psi_R^8 = \begin{pmatrix}0 & \widetilde{R}^8\end{pmatrix};\ \Psi_L^8 = \begin{pmatrix}0 & \widetilde{L}^8\end{pmatrix}. \tag{2.89}$$

With P_0 we have

$$\begin{aligned}
&P_0(\Psi_R^8) = 2p\Psi_R^8\gamma_{21};\ \exp(a^0P_0)(\Psi_R^8) = \Psi_R^8\exp[2pa^0\gamma_{21}],\\
&P_0(\Psi_R^1) = 2\Psi_R^1\gamma_{21};\ \exp(a^0P_0)(\Psi_R^1) = \Psi_R^1\exp[2a^0\gamma_{21}],\\
&P_0(\Psi_L) = \Psi_L\gamma_{21};\ \exp(a^0P_0)(\Psi_L) = \Psi_L\exp[a^0\gamma_{21}].
\end{aligned} \tag{2.90}$$

Next we let:

$$\begin{aligned}
&s = \theta u = \theta(s_1 P_1 + s_2 P_2 + s_3 P_3);\ u^2 = s_1^2 + s_2^2 + s_3^2 = 1, \\
&U = e^s = e^{\theta u},
\end{aligned} \tag{2.91}$$

and we have:

$$\begin{aligned}
\Psi_R'^1 &= U\Psi_R^1 = \Psi_R^1;\ \Psi_R'^8 = U\Psi_R^8 = \Psi_R^8, \\
\Psi_L' &= U\Psi_L = \cos(\theta)\Psi_L + \sin(\theta)u(\Psi_L), \\
u(\Psi_L) &= s_1\Psi_L\gamma_3\mathbf{i} + s_2\Psi_L\gamma_3 + s_3\Psi_L(-\mathbf{i}), \\
\Psi_L &= U^{-1}\Psi_L' = \cos(\theta)\Psi_L' - \sin(\theta)u(\Psi_L') \\
&= \cos(\theta)\Psi_L' - \sin(\theta)[s_1\Psi_L'\gamma_3\mathbf{i} + s_2\Psi_L'\gamma_3 + s_3\Psi_L'(-\mathbf{i})].
\end{aligned} \tag{2.92}$$

Since P_0 commutes with s we have:

$$\exp(S) = \exp(a^0 P_0)e^s = e^s \exp(a^0 P_0);\ \exp(-S) = \exp(S)^{-1}. \tag{2.93}$$

The set of the $\exp(S)$ is a $U(1) \times SU(2)$ Lie group. The local gauge transformation uses the derivative of the exponential function and satisfies

$$\Psi' = [\exp(S)](\Psi)\ ;\ \mathrm{D} = \sigma^\mu D_\mu\ ;\ \mathrm{D}' = \sigma^\mu D_\mu', \tag{2.94}$$

and so $D_\mu\Psi$ is replaced by $D_\mu'\Psi'$ such that:

$$(0\ \ \mathrm{D}')\Psi' = (0\ \ \nabla)\Psi' + G'(\Psi') = \exp(S)[(0\ \ \nabla)\Psi + G(\Psi)], \tag{2.95}$$

$$G'(\Psi') = \exp(S)(X+Y);\ X = \Big[(0\ \ \nabla)[\exp(-S)]\Big](\Psi');\ Y = G(\Psi). \tag{2.96}$$

The transformation of the gauge potentials thus has two parts: a part that comes from the derivative of the exponential function and another that comes from the non-commutation of $\exp(S)$ with P_j.

2.3.1 The $U(1)$ gauge group generated by P_0

Since P_0 commutes with s, the relation between $\mathrm{w}'^0 = \mathrm{b}'$ and $\mathrm{w}^0 = \mathrm{b}$ is reduced to only the part coming from the derivative, and we get:

$$\mathrm{b}_\mu' = \mathrm{b}_\mu - \partial_\mu a^0. \tag{2.97}$$

The different space-time vectors that we may form from the wave with spin 1/2 to obtain the gauge potentials w^j, where $j = 0, 1, 2, 3$, in a similitude must behave like J_l which is the sum of D_R^1, D_R^8, D_L^1 and D_L^8. In addition to these vectors we have:

$$\begin{aligned}
\mathrm{D}_{RL}^1 &= \mathrm{D}_1 = R^1\sigma_1\widetilde{L}^1 + L^1\sigma_1\widetilde{R}^1;\ d_{RL}^1 = \mathrm{D}_2 = i(R^1\sigma_1\widetilde{L}^1 - L^1\sigma_1\widetilde{R}^1),\\
\mathrm{D}_L^{18} &= L^1L^8 + \widetilde{L}^8\widetilde{L}^1;\ d_L^{18} = i(L^1L^8 - \widetilde{L}^8\widetilde{L}^1),\\
\mathrm{D}_{RL}^{18} &= R^1\sigma_1L^8 + \widetilde{L}^8\sigma_1\widetilde{R}^1;\ d_{RL}^{18} = i(R^1\sigma_1L^8 - \widetilde{L}^8\sigma_1\widetilde{R}^1),\\
\mathrm{D}_R^{18} &= R^1R^8 + \widetilde{R}^8\widetilde{R}^1;\ d_R^{18} = i(R^1R^8 - \widetilde{R}^8\widetilde{R}^1),\\
\mathrm{D}_{RL}^{81} &= \widetilde{R}^8\sigma_1\widetilde{L}^1 + L^1\sigma_1R^8;\ d_{RL}^{81} = i(\widetilde{R}^8\sigma_1\widetilde{L}^1 + L^1\sigma_1R^8),\\
\mathrm{D}_{RL}^8 &= \widetilde{R}^8\sigma_1L^8 + \widetilde{L}^8\sigma_1R^8;\ d_{RL}^8 = i(\widetilde{R}^8\sigma_1L^8 - \widetilde{L}^8\sigma_1R^8).
\end{aligned} \tag{2.98}$$

All these D vectors transform in a similitude defined by M into $\mathrm{D}' = M\mathrm{D}\widetilde{M}$, like J_l. With (2.90) we have:

$$\begin{aligned}
&\widetilde{L}'^8 = e^{-ia^0}\widetilde{L}^8;\ \widetilde{R}'^8 = e^{2ipa^0}\widetilde{R}^8;\ L'^1 = e^{-ia^0}L^1;\ R'^1 = e^{2ia^0}R^1, &(2.99)\\
&\widetilde{R}'^{\overline{8}} = e^{ia^0}\widetilde{R}^{\overline{8}};\ \widetilde{L}'^{\overline{8}} = e^{-2ipa^0}\widetilde{L}^{\overline{8}};\ R'^{\overline{1}} = e^{ia^0}R^{\overline{1}};\ L'^{\overline{1}} = e^{-2ia^0}L^{\overline{1}}.
\end{aligned}$$

This gives

$$\begin{aligned}
R'^1\sigma_1\widetilde{L}'^1 &= e^{3ia^0}R^1\sigma_1\widetilde{L}^1;\ R'^1\sigma_1L'^8 = e^{3ia^0}R^1\sigma_1L^8,\\
L'^1L'^8 &= L^1L^8;\ \widetilde{R}'^8\sigma_1\widetilde{L}'^1 = e^{i(1+2p)a^0}\widetilde{R}^8\sigma_1\widetilde{L}^1,\\
R'^1R'^8 &= e^{2i(1-p)a^0}R^1R^8;\ \widetilde{R}'^8\sigma_1L'^8 = e^{i(1+2p)a^0}\widetilde{R}^8\sigma_1L^8.
\end{aligned} \tag{2.100}$$

We also have

$$\mathrm{D}_R'^1 = R'^1\widetilde{R}'^1 = e^{2ia^0}R^1e^{-2ia^0}\widetilde{R}^1 = R^1\widetilde{R}^1 = \mathrm{D}_R^1, \tag{2.101}$$

and similarly:

$$\mathrm{D}_R'^8 = \mathrm{D}_R^8;\ \mathrm{D}_L'^1 = \mathrm{D}_L^1;\ \mathrm{D}_L'^8 = \mathrm{D}_L^8;\ \mathrm{D}_R'^8 = \mathrm{D}_R^8;\ \mathrm{J}_l' = \mathrm{J}_l;\ \mathrm{v}' = \mathrm{v}. \tag{2.102}$$

We then get:

$$\begin{aligned}
\widehat{\nabla}R'^1 &= \widehat{\nabla}(e^{2ia^0}R^1) = e^{2ia^0}[2i(\widehat{\nabla}a^0)R^1 + \widehat{\nabla}R^1]\\
&= e^{2ia^0}[2i(\widehat{\nabla}a^0)R^1 - i(2\widehat{\mathrm{b}} + m\widehat{\mathrm{v}})R^1]\\
&= i[2e^{2ia^0}(\widehat{\nabla}a^0 - \widehat{\mathrm{b}}) - m\widehat{\mathrm{v}}e^{2ia^0}]R^1 = -i(2\widehat{\mathrm{b}}' + m\widehat{\mathrm{v}})R'^1, &(2.103)\\
\widehat{\mathrm{b}}' &= \widehat{\mathrm{b}} - \widehat{\nabla}a^0. &(2.104)
\end{aligned}$$

The gauge invariance with P_0 of the other parts of the leptonic wave acts in the same way; this is the case with (2.104) as well as for Ψ_R^8 or Ψ_R^1 or Ψ_L^n.

2.3.2 The $SU(2)$ gauge group

This gauge group acts only on the left part of the waves in the lepton case. Then we only need to think about $\Psi_L = \Psi_L^1 + \Psi_L^8$. The gauge transformation reads:

$$\Psi'_L = U(\Psi_L) = \cos(\theta)\Psi_L + \sin(\theta)[s_1\Psi_L\gamma_3 + s_2\Psi_L\gamma_3(-\mathbf{i}) + s_3\Psi_L(-\mathbf{i})]. \tag{2.105}$$

The gauge invariance means that with

$$\mathbf{D} = (0\ \ \mathrm{D}) = \boldsymbol{\partial} + \mathbf{G};\ \boldsymbol{\partial} = (0\ \ \nabla);\ (0\ \ \mathrm{D}') = \mathbf{D}' = \boldsymbol{\partial} + \mathbf{G}', \tag{2.106}$$

we must have

$$\mathbf{D}'\Psi'_L = U(\mathbf{D}\Psi_L), \tag{2.107}$$

which necessitates:

$$\mathbf{G}'(\Psi'_L) = U(X) + U(Y);\ X = [\boldsymbol{\partial}(U^{-1})](\Psi'_L);\ Y = \mathbf{G}(\Psi_L) \tag{2.108}$$

$$X = \boldsymbol{\partial}(\cos\theta)\Psi'_L - [\boldsymbol{\partial}(s_1\sin\theta)\Psi'_L\gamma_3 + \boldsymbol{\partial}(s_2\sin\theta)\Psi'_L\gamma_3(-\mathbf{i})$$
$$+ \boldsymbol{\partial}(s_3\sin\theta)\Psi'_L(-\mathbf{i})].$$

We then get:

$$\mathbf{w}^j = (0\ \ \mathrm{w}^j);\ j = 1,2,3, \tag{2.109}$$

$$\begin{aligned} U(X) = &-[s_1\boldsymbol{\partial}\theta + \frac{\sin(2\theta)}{2}\boldsymbol{\partial}s_1 + \sin^2(\theta)(s_2\boldsymbol{\partial}s_3 - s_3\boldsymbol{\partial}s_2)]\Psi'_L\gamma_3 \\ &- [s_2\boldsymbol{\partial}\theta + \frac{\sin(2\theta)}{2}\boldsymbol{\partial}s_2 + \sin^2(\theta)(s_3\boldsymbol{\partial}s_1 - s_1\boldsymbol{\partial}s_3)]\Psi'_L\gamma_3(-\mathbf{i}) \\ &- [s_3\boldsymbol{\partial}\theta + \frac{\sin(2\theta)}{2}\boldsymbol{\partial}s_3 + \sin^2(\theta)(s_1\boldsymbol{\partial}s_2 - s_2\boldsymbol{\partial}s_1)]\Psi'_L(-\mathbf{i}). \end{aligned} \tag{2.110}$$

$$\begin{aligned} U(Y) = &\cos(2\theta)[\mathbf{w}^1\Psi'_L\gamma_3 + \mathbf{w}^2\Psi'_L\gamma_3(-\mathbf{i}) + \mathbf{w}^3\Psi'_L(-\mathbf{i})] + \sin(2\theta) \\ &\times [(s_2\mathbf{w}^3 - s_3\mathbf{w}^2)\Psi'_L\gamma_3\mathbf{i} + (s_3\mathbf{w}^1 - s_1\mathbf{w}^3)\Psi'_L\gamma_3(-\mathbf{i}) \\ &+ (s_1\mathbf{w}^2 - s_2\mathbf{w}^1)\Psi'_L(-\mathbf{i})] \\ &+ 2\sin^2(\theta)(s_1\mathbf{w}^1 + s_2\mathbf{w}^2 + s_3\mathbf{w}^3)[s_1\Psi'_L\gamma_3 + s_2\Psi'_L\gamma_3(-\mathbf{i}) + s_3\Psi'_L(-\mathbf{i})]. \end{aligned} \tag{2.111}$$

Hence we finally have

$$\begin{aligned} \mathbf{w}'^1 = &-[s_1\boldsymbol{\partial}\theta + \frac{\sin(2\theta)}{2}\boldsymbol{\partial}s_1 + \sin^2(\theta)(s_2\boldsymbol{\partial}s_3 - s_3\boldsymbol{\partial}s_2)] \\ &+ \cos(2\theta)\mathbf{w}^1 + \sin(2\theta)(s_2\mathbf{w}^3 - s_3\mathbf{w}^2) + 2\sin^2(\theta)s_1(s_1\mathbf{w}^1 + s_2\mathbf{w}^2 + s_3\mathbf{w}^3), \\ \mathbf{w}'^2 = &-[s_2\boldsymbol{\partial}\theta + \frac{\sin(2\theta)}{2}\boldsymbol{\partial}s_2 + \sin^2(\theta)(s_3\boldsymbol{\partial}s_1 - s_1\boldsymbol{\partial}s_3)] \\ &+ \cos(2\theta)\mathbf{w}^2 + \sin(2\theta)(s_3\mathbf{w}^1 - s_1\mathbf{w}^3) + 2\sin^2(\theta)s_2(s_1\mathbf{w}^1 + s_2\mathbf{w}^2 + s_3\mathbf{w}^3), \\ \mathbf{w}'^3 = &-[s_3\boldsymbol{\partial}\theta + \frac{\sin(2\theta)}{2}\boldsymbol{\partial}s_3 + \sin^2(\theta)(s_1\boldsymbol{\partial}s_2 - s_2\boldsymbol{\partial}s_1)] \\ &+ \cos(2\theta)\mathbf{w}^3 + \sin(2\theta)(s_1\mathbf{w}^2 - s_2\mathbf{w}^1) + 2\sin^2(\theta)s_3(s_1\mathbf{w}^1 + s_2\mathbf{w}^2 + s_3\mathbf{w}^3). \end{aligned} \tag{2.112}$$

We may note the 3-order symmetry of these equalities: $SU(2)$ is a 3-dimensional Lie group. This symmetry is no longer an "internal" symmetry but an invariance under a geometrical group emerging from the properties of multiplication in the Cl_3 algebra.

Gauge generated by P_3

We arrive at the one-parameter group generated by P_3 with $a^0 = s_1 = s_2 = 0$ and $s_3 = 1$. We thus get:

$$\begin{aligned} S &= s = \theta P_3, \\ \Psi' &= [\exp(S)](\Psi) = P_-(\Psi) + \cos(\theta) P_+(\Psi) + \sin(\theta) P_3(\Psi) \\ &= \begin{pmatrix} R^1 + e^{-i\theta} L^1 & \widetilde{R}^8 + e^{i\theta}\widetilde{L}^8 \end{pmatrix}. \end{aligned} \tag{2.113}$$

So we have:

$$\begin{aligned} &R'^1 = R^1;\ \widetilde{R}'^8 = \widetilde{R}^8;\ L'^1 = e^{-i\theta} L^1;\ \widetilde{L}'^8 = e^{i\theta}\widetilde{L}^8;\ \ \mathrm{J}'_l = \mathrm{J}_l, \\ &L'^{\overline{1}} = L^{\overline{1}};\ \widetilde{L}'^{\overline{8}} = \widetilde{L}^{\overline{8}};\ R'^{\overline{1}} = e^{-i\theta} R^{\overline{1}};\ \widetilde{R}'^{\overline{8}} = e^{i\theta}\widetilde{R}^{\overline{8}}. \end{aligned} \tag{2.114}$$

We also have $\mathrm{w}'^0_\mu = \mathrm{w}^0_\mu$, which means $\mathrm{b}' = \mathrm{b}$. The equations (2.112) become

$$\begin{aligned} \mathrm{w}'^1 &= \cos(2\theta)\mathrm{w}^1 - \sin(2\theta)\mathrm{w}^2, \\ \mathrm{w}'^2 &= \cos(2\theta)\mathrm{w}^2 + \sin(2\theta)\mathrm{w}^1, \\ \mathrm{w}'^3 &= -i\nabla\theta + \mathrm{w}^3. \end{aligned} \tag{2.115}$$

And we have

$$\begin{aligned} \mathrm{D}'^{18}_L - i d'^{18}_L &= 2L'^1 L'^8 = 2e^{-i\theta} L^1 e^{-i\theta} L^8 = e^{-2i\theta}(\mathrm{D}^{18}_L - i d^{18}_L) \\ &= \cos(2\theta)\mathrm{D}^{18}_L - \sin(2\theta) d^{18}_L - i[\sin(2\theta)\mathrm{D}^{18}_L + \cos(2\theta) d^{18}_L], \\ \mathrm{D}'^{18}_L &= \cos(2\theta)\mathrm{D}^{18}_L - \sin(2\theta) d^{18}_L, \\ d'^{18}_L &= \sin(2\theta)\mathrm{D}^{18}_L + \cos(2\theta) d^{18}_L. \end{aligned} \tag{2.116}$$

This is compatible with

$$W^1 = \mathbf{k}(\mathrm{D}^{18}_L);\ W^2 = \mathbf{k}(d^{18}_L);\ \mathrm{w}^1 = \frac{g_2}{2}\mathbf{k}(\mathrm{D}^{18}_L);\ \mathrm{w}^2 = \frac{g_2}{2}\mathbf{k}(d^{18}_L), \tag{2.117}$$

where $\mathbf{k}$ is any linear operator. The g_2 coefficient called the "coupling constant" is necessary for transforming the contravariant vector W^j into the covariant vector w_j (see 1.7).

Gauge generated by P_1

We now have $a^0 = s_2 = s_3 = 0$ and $s_1 = 1$. We then have:

$$\begin{aligned} S &= s = \theta P_1, \\ \Psi' &= [\exp(S)](\Psi) = P_-(\Psi) + \cos(\theta) P_+(\Psi) + \sin(\theta) P_1(\Psi) \\ &= \begin{pmatrix} R^1 + \cos(\theta) L^1 + i\sin(\theta)\widetilde{L}^8 & \widetilde{R}^8 + \cos(\theta)\widetilde{L}^8 + i\sin(\theta) L^1 \end{pmatrix}. \end{aligned} \tag{2.118}$$

So we get:

$$
\begin{aligned}
&R'^1 = R^1;\ \ R'^8 = R^8;\ L'^{\overline{1}} = L^{\overline{1}};\ L'^{\overline{8}} = L^{\overline{8}},\\
&L'^1 = \cos(\theta)L^1 + i\sin(\theta)\widetilde{L}^8;\ R'^{\overline{1}} = \cos(\theta)R^{\overline{1}} + i\sin(\theta)\widetilde{R}^{\overline{8}},\\
&\widetilde{L}'^8 = \cos(\theta)\widetilde{L}^8 + i\sin(\theta)L^1;\ \widetilde{R}'^{\overline{8}} = \cos(\theta)\widetilde{R}^{\overline{8}} + i\sin(\theta)R^{\overline{1}}.
\end{aligned}
\tag{2.119}
$$

Hence we have:

$$
\begin{aligned}
\mathrm{D}'^1_L &= L'^1\widetilde{L}'^1 = [\cos(\theta)L^1 + i\sin(\theta)\widetilde{L}^8][\cos(\theta)\widetilde{L}^1 - i\sin(\theta)L^8]\\
&= \cos^2(\theta)L^1\widetilde{L}^1 + i\sin(\theta)\cos(\theta)(\widetilde{L}^8\widetilde{L}^1 - L^1L^8) + \sin^2(\theta)\widetilde{L}^8L^8,\\
\mathrm{D}'^8_L &= \widetilde{L}'^8L'^8 = [\cos(\theta)\widetilde{L}^8 + i\sin(\theta)L^1][\cos(\theta)L^8 - i\sin(\theta)\widetilde{L}^1] \qquad (2.120)\\
&= \sin^2(\theta)L^1\widetilde{L}^1 - i\sin(\theta)\cos(\theta)(\widetilde{L}^8\widetilde{L}^1 - L^1L^8) + \cos^2(\theta)\widetilde{L}^8L^8.
\end{aligned}
$$

We derive the following:

$$
\mathrm{D}'^1_L + \mathrm{D}'^8_L = \mathrm{D}^1_L + \mathrm{D}^8_L;\ \ \mathrm{J}'_l = \mathrm{J}_l;\ \rho'_l = \rho_l;\ \mathrm{v}' = \mathrm{v}. \tag{2.121}
$$

The mass term is thus invariant under the gauge transformation. We also derive from (2.120):

$$
\mathrm{D}'^8_L - \mathrm{D}'^1_L = \cos(2\theta)(\mathrm{D}^8_L - \mathrm{D}^1_L) + \sin(2\theta)d^{18}_L. \tag{2.122}
$$

Next we have:

$$
\begin{aligned}
2L'^1L'^8 &= \mathrm{D}'^{18}_L - id'^{18}_L \qquad (2.123)\\
&= 2[\cos(\theta)L^1 + i\sin(\theta)\widetilde{L}^8][\cos(\theta)L^8 - i\sin(\theta)\widetilde{L}^1]\\
&= \mathrm{D}^{18}_L - i[\cos(2\theta)d^{18}_L - \sin(2\theta)(\mathrm{D}^8_L - \mathrm{D}^1_L)].
\end{aligned}
$$

The equations (2.112) become:

$$
\begin{aligned}
\mathrm{w}'^2 &= \cos(2\theta)\mathrm{w}^2 - \sin(2\theta)\mathrm{w}^3,\\
\mathrm{w}'^3 &= \cos(2\theta)\mathrm{w}^3 + \sin(2\theta)\mathrm{w}^2,\\
\mathrm{w}'^1 &= -\nabla\theta + \mathrm{w}^1.
\end{aligned}
\tag{2.124}
$$

All this is then compatible with:

$$
W^3 = \mathbf{k}(\mathrm{D}^8_L - \mathrm{D}^1_L). \tag{2.125}
$$

Gauge generated by P_2

We now have $a^0 = s_1 = s_3 = 0$ and $s_2 = 1$. We then get:

$$
\Psi_L = \begin{pmatrix} L^1 & \widetilde{L}^8 \end{pmatrix};\ P_2(\Psi_L) = \Psi_L\gamma_3 = \begin{pmatrix} \widetilde{L}^8 & -L^1 \end{pmatrix}, \tag{2.126}
$$

$$
\begin{aligned}
S &= s = \theta P_2,\\
\Psi' &= [\exp(S)](\Psi) = P_-(\Psi) + \cos(\theta)P_+(\Psi) + \sin(\theta)P_2(\Psi) \qquad (2.127)\\
&= \begin{pmatrix} R^1 + \cos(\theta)L^1 + \sin(\theta)\widetilde{L}^8 & \widetilde{R}^8 + \cos(\theta)\widetilde{L}^8 - \sin(\theta)L^1 \end{pmatrix}.
\end{aligned}
$$

Hence we have:

$$\begin{aligned}
&R'^1 = R^1;\quad R'^8 = R^8,\ \mathrm{D}'^1_R = \mathrm{D}^1_R;\quad \mathrm{D}'^8_R = \mathrm{D}^8_R;\quad L'^{\overline{1}} = L^{\overline{1}};\quad L'^{\overline{8}} = L^{\overline{8}},\\
&L'^1 = \cos(\theta)L^1 + \sin(\theta)\widetilde{L}^8;\quad R'^{\overline{1}} = \cos(\theta)R^{\overline{1}} - \sin(\theta)\widetilde{R}^{\overline{8}}, \qquad (2.128)\\
&\widetilde{L}'^8 = \cos(\theta)\widetilde{L}^8 - \sin(\theta)L^1;\quad \widetilde{R}'^{\overline{8}} = \cos(\theta)\widetilde{R}^{\overline{8}} + \sin(\theta)R^{\overline{1}}.
\end{aligned}$$

We may notice that the changes of sign when we pass from the wave of the particle to the wave of the antiparticle are the origin of what we saw in 2.1.2: the charge conjugation changes the rotation of the matrix indices for the doublet of right waves. We then get:

$$\begin{aligned}
\mathrm{D}'^1_L &= L'^1\widetilde{L}'^1 = [\cos(\theta)L^1 + \sin(\theta)\widetilde{L}^8][\cos(\theta)\widetilde{L}^1 + \sin(\theta)L^8]\\
&= \cos^2(\theta)L^1\widetilde{L}^1 + \sin(\theta)\cos(\theta)(\widetilde{L}^8\widetilde{L}^1 + L^1L^8) + \sin^2(\theta)\widetilde{L}^8L^8,\\
\mathrm{D}'^8_L &= \widetilde{L}'^8L'^8 = [\cos(\theta)\widetilde{L}^8 - \sin(\theta)L^1][\cos(\theta)L^8 - \sin(\theta)\widetilde{L}^1] \qquad (2.129)\\
&= \sin^2(\theta)L^1\widetilde{L}^1 - \sin(\theta)\cos(\theta)(\widetilde{L}^8\widetilde{L}^1 + L^1L^8) + \cos^2(\theta)\widetilde{L}^8L^8.
\end{aligned}$$

We arrive at the following:

$$\mathrm{D}'^1_L + \mathrm{D}'^8_L = \mathrm{D}^1_L + \mathrm{D}^8_L;\ \mathrm{J}'_l = \mathrm{J}_l. \qquad (2.130)$$

We derive also from (2.129):

$$\mathrm{D}'^8_L - \mathrm{D}'^1_L = \cos(2\theta)(\mathrm{D}^8_L - \mathrm{D}^1_L) - \sin(2\theta)\mathrm{D}^{18}_L. \qquad (2.131)$$

Next we have:

$$\begin{aligned}
2L'^1L'^8 &= \mathrm{D}'^{18}_L - id'^{18}_L \qquad (2.132)\\
&= 2[\cos(\theta)L^1 + \sin(\theta)\widetilde{L}^8][\cos(\theta)L^8 - \sin(\theta)\widetilde{L}^1]\\
&= -id^{18}_L + [\cos(2\theta)\mathrm{D}^{18}_L + \sin(2\theta)(\mathrm{D}^8_L - \mathrm{D}^1_L)].
\end{aligned}$$

The equations (2.112) become:

$$\begin{aligned}
\mathrm{w}'^3 &= \cos(2\theta)\mathrm{w}^3 - \sin(2\theta)\mathrm{w}^1,\\
\mathrm{w}'^1 &= \cos(2\theta)\mathrm{w}^1 + \sin(2\theta)\mathrm{w}^3,\\
\mathrm{w}'^2 &= -\nabla\theta + \mathrm{w}^2. \qquad (2.133)
\end{aligned}$$

This is compatible with (2.117) and (2.125).

The very short range of the weak interaction was thought to be linked to a strong mass of the potential vectors. The expected wave equations were:

$$(\Box + m^2)W^k = 0;\ (\Box + m'^2)Z^0 = 0. \qquad (2.134)$$

The link between potential A, field F and electric current j is, in the electromagnetic case:

$$F = \nabla\widehat{A};\ \nabla\widehat{F} = \mathrm{j};\ \Box A = \nabla\widehat{\nabla}A = \nabla\widehat{F} = \mathrm{j}. \qquad (2.135)$$

The relation between potential X and current $D_L^8 - D_L^1$ may be:

$$0 = (\Box + m^2)X;\ \Box X = -m^2 X = D_L^8 - D_L^1; \tag{2.136}$$

We then may use:

$$W^3 = \mathbf{k}(\mathrm{D}_L^8 - \mathrm{D}_L^1);\quad W^1 = \mathbf{k}(\mathrm{D}_L^{18});\quad W^2 = \mathbf{k}(d_L^{18}). \tag{2.137}$$

We may also introduce here a constant $\mathbf{k} = 1$ since the constant g_2 is already a factor of W^j in the wave equations. Hence the simplest form of the previous relations is $\mathbf{k} = \mathrm{id}$:

$$W^3 = \mathrm{D}_L^8 - \mathrm{D}_L^1;\quad W^1 = \mathrm{D}_L^{18};\quad W^2 = d_L^{18}. \tag{2.138}$$

2.3.3 Simplification of the equations

Since W^3 and B are, like J_l, linear combinations of the chiral currents D_R^1, D_R^8, D_L^1 and D_L^8, and since $\widetilde{L}^1\widehat{L}^1 = 0$, we have with (2.137):

$$\begin{aligned}(W^1 + iW^2)\overline{L}^8 - W^3\widehat{L}^1 &= i[2\widetilde{L}^8\widetilde{L}^1\overline{L}^8 - (\widetilde{L}^8 L^8 - L^1\widetilde{L}^1)\widehat{L^1}]\\ &= i[\widetilde{L}^8(2\widetilde{L}^1\overline{L}^8 - L^8\widehat{L}^1)].\end{aligned} \tag{2.139}$$

And we have:

$$\begin{aligned}\widetilde{L}^1\overline{L}^8 &= 2\begin{pmatrix}0 & 0\\ -\eta_2^1 & \eta_1^1\end{pmatrix}\begin{pmatrix}\eta_1^8 & 0\\ \eta_2^8 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ -a_2 & 0\end{pmatrix} = -a_2\frac{\sigma_1 - i\sigma_2}{2},\\ L^8\widehat{L}^1 &= \overline{\widetilde{L}^1\overline{L}^8} = a_2\frac{\sigma_1 - i\sigma_2}{2} = -\widetilde{L}^1\overline{L}^8,\\ (W^1 + iW^2)\overline{L}^8 - W^3\widehat{L}^1 &= -3\widetilde{L}^8 L^8\widehat{L}^1 = -3\mathrm{D}_L^8\widehat{L}^1\\ &= -3(\mathrm{D}_L^8 - \mathrm{D}_L^1)\widehat{L}^1 = -3W^3\widehat{L}^1.\end{aligned} \tag{2.140}$$

And similarly we arrive at

$$\begin{aligned}(W^1 - iW^2)\widehat{L}^1 + W^3\overline{L}^8 &= 2L^1L^8\widehat{L}^1 + (\widetilde{L}^8L^8 - L^1\widetilde{L}^1)\overline{L}^8\\ &= [2L^1L^8\widehat{L}^1 - L^1\widetilde{L}^1\overline{L}^8] = -3L^1\widetilde{L}^1\overline{L}^8 = -3\mathrm{D}_L^1\overline{L}^8\\ &= 3(\widetilde{L}^8L^8 - L^1\widetilde{L}^1)\overline{L}^8 = 3W^3\overline{L}^8.\end{aligned} \tag{2.141}$$

Then the equations of the left waves can be expressed as follows, by simplifying the three terms W^j:

$$0 = (\nabla + i\mathrm{b} + 3i\mathrm{w}^3 + i\mathbf{l}\mathrm{v})\widehat{L}^1;\ 0 = (\nabla + i\mathrm{b} - 3i\mathrm{w}^3 + im_l\mathrm{v})\widehat{L}^8. \tag{2.142}$$

The four spinor wave equations become

$$i\nabla\eta^1 = (\mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v})\eta^1;\ i\widehat{\nabla}\xi^1 = (2\widehat{\mathrm{b}} + \mathbf{r}\widehat{\mathrm{v}})\xi^1, \tag{2.143}$$

$$i\widetilde{\nabla}\eta^8 = (\mathrm{b} - 3\mathrm{w}^3 + m_l\mathrm{v})\eta^8;\ i\overline{\nabla}\xi^8 = (2p\widehat{\mathrm{b}} + m_5\widehat{\mathrm{v}})\xi^8. \tag{2.144}$$

With the l^n and r^n in (2.10) until (2.13), this corresponds to:

$$\mathrm{a}^1 = \mathrm{b} + 3\mathrm{w}^3;\ l^1 = \mathrm{a}^1 + \mathbf{l}\mathrm{v} = \mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v};\ 0 = -i\nabla\eta^1 + l^1\eta^1, \tag{2.145}$$

$$\mathrm{a}^2 = 2\mathrm{b};\ r^1 = \mathrm{a}^2 + \mathbf{r}\mathrm{v} = 2\mathrm{b} + \mathbf{r}\mathrm{v};\ 0 = -i\widehat{\nabla}\xi^1 + \widehat{r}^1\xi^1, \tag{2.146}$$

$$\mathrm{a}^3 = \mathrm{b} - 3\mathrm{w}^3;\ l^8 = \mathrm{a}^3 + m_l\mathrm{v} = \mathrm{b} - 3\mathrm{w}^3 + m_l\mathrm{v};\ 0 = -i\nabla\eta^8 + l^8\eta^8, \tag{2.147}$$

$$\mathrm{a}^4 = 2p\mathrm{b};\ r^8 = \mathrm{a}^4 + m_r\mathrm{v} = 2p\mathrm{b} + m_r\mathrm{v};\ 0 = -i\widehat{\nabla}\xi^8 + \widehat{r}^8\xi^8. \tag{2.148}$$

2.3.4 Double link with the Lagrangian density

From the left side we multiply (2.145) by $\eta^{1\dagger}$, (2.146) by $\xi^{1\dagger}$, (2.147) by $\eta^{8\dagger}$ and (2.148) by $\xi^{8\dagger}$:

$$0 = \mathcal{L}^1 = -i\eta^{1\dagger}\nabla\eta^1 + \eta^{1\dagger}l^1\eta^1, \tag{2.149}$$

$$0 = \mathcal{L}^2 = -i\xi^{1\dagger}\widehat{\nabla}\xi^1 + \xi^{1\dagger}\widehat{r}^1\xi^1, \tag{2.150}$$

$$0 = \mathcal{L}^3 = -i\eta^{8\dagger}\widetilde{\nabla}\eta^8 + \eta^{8\dagger}l^8\eta^8, \tag{2.151}$$

$$0 = \mathcal{L}^4 = -i\xi^{8\dagger}\overline{\nabla}\xi^8 + \xi^{8\dagger}\widehat{r}^8\xi^8, \tag{2.152}$$

$$0 = \mathcal{L} = \frac{m}{k\mathbf{l}}\mathcal{L}^1 + \frac{m}{k\mathbf{r}}\mathcal{L}^2 + \frac{m}{km_l}\mathcal{L}^3 + \frac{m}{km_r}\mathcal{L}^4. \tag{2.153}$$

From this construction the Lagrangian density $\mathcal{L}$ is stationary, since it is identically null at any point in space-time, and not only on average. No physical principle is used to obtain this result: the Lagrangian density is null as a sum of null terms. **The principle of least action is no longer a quasi-metaphysical principle**. We will read each of the terms of this Lagrangian density as the sum of a real part and an imaginary part. We may first remark that we will repeat the same procedure four times [7], and thus it is enough to completely work out the only $\mathcal{L}^1$ part. We have:

$$\begin{aligned}
\mathcal{L}^1 &= \frac{1}{2}(\mathcal{L}^1 + \mathcal{L}^{1\dagger}) + \frac{1}{2}(\mathcal{L}^1 - \mathcal{L}^{1\dagger}),\\
\mathcal{L}^1 + \mathcal{L}^{1\dagger} &= -i\eta^{1\dagger}\sigma^\mu(\partial_\mu\eta^1) + \eta^{1\dagger}l^1\eta^1 + i(\partial_\mu\eta^{1\dagger})\sigma^\mu\eta^1 + \eta^{1\dagger}l^{1\dagger}\eta^1\\
&= -i\eta^{1\dagger}\sigma^\mu(\partial_\mu\eta^1) + i(\partial_\mu\eta^{1\dagger})\sigma^\mu\eta^1 + \eta^{1\dagger}(l^1 + l^{1\dagger})\eta^1. \qquad (2.154)\\
\mathcal{L}^1 - \mathcal{L}^{1\dagger} &= -i\eta^{1\dagger}\sigma^\mu(\partial_\mu\eta^1) + \eta^{1\dagger}l^1\eta^1 - i(\partial_\mu\eta^{1\dagger})\sigma^\mu\eta^1 - \eta^{1\dagger}l^{1\dagger}\eta^1\\
&= -i\partial_\mu(\eta^{1\dagger}\sigma^\mu\eta^1) + \eta^{1\dagger}(l^1 - l^{1\dagger})\eta^1.
\end{aligned}$$

7. When we pass from η to ξ we replace σ^μ with $\widehat{\sigma}^\mu$, and so we have only three signs to change; otherwise all is similar.

Each l^n and r^n is a sum of vectors, and space-time vectors form the self-adjoint part of the space algebra. We then have:

$$\begin{aligned}
\frac{1}{2}(l^1 + l^{1\dagger}) &= l^1 = l^1_\mu \sigma^\mu;\ \frac{1}{2}(l^1 - l^{1\dagger}) = 0,\\
0 = \frac{1}{2}(\mathcal{L}^1 + \mathcal{L}^{1\dagger}) &= \frac{1}{2}[-i\eta^{1\dagger}\sigma^\mu(\partial_\mu \eta^1) + i(\partial_\mu \eta^{1\dagger})\sigma^\mu \eta^1] + \eta^{1\dagger} l^1 \eta^1, \quad (2.155)\\
0 = \frac{1}{2}(\mathcal{L}^1 - \mathcal{L}^{1\dagger}) &= \frac{1}{2}[-i\eta^{1\dagger}\sigma^\mu(\partial_\mu \eta^1) - i(\partial_\mu \eta^{1\dagger})\sigma^\mu \eta^1] = -\frac{i}{2}\partial_\mu \mathrm{D}^{1\mu}_L.
\end{aligned}$$

This last relation means that the D^1_L current is conservative. And since the three other equations behave similarly, the D^1_R, D^8_L and D^8_R currents are also conservative. Therefore the $\mathcal{L}^n$ terms which have a null imaginary part are real, and thus equal to their real part:

$$0 = \mathcal{L}^n = \Re(\mathcal{L}^n). \quad (2.156)$$

Now we completely calculate this equation, arriving at real numbers with the help of the following real matrix representation:

$$\eta^1 = \begin{pmatrix} a+ib \\ c+id \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \\ c & -d \\ d & c \end{pmatrix}, \quad (2.157)$$

$$\eta^{1\dagger} = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \end{pmatrix};\ -i\eta^{1\dagger} = \begin{pmatrix} -b & a & -d & c \\ -a & -b & -c & -d \end{pmatrix}, \quad (2.158)$$

$$\nabla = \sigma^\mu \partial_\mu = \begin{pmatrix} \partial_0 - \partial_3 & 0 & -\partial_1 & -\partial_2 \\ 0 & \partial_0 - \partial_3 & \partial_2 & -\partial_1 \\ -\partial_1 & \partial_2 & \partial_0 + \partial_3 & 0 \\ -\partial_2 & -\partial_1 & 0 & \partial_0 + \partial_3 \end{pmatrix}, \quad (2.159)$$

$$l^1 = \begin{pmatrix} l^{10} + l^{13} & 0 & l^{11} & l^{12} \\ 0 & l^{10} + l^{13} & -l^{12} & l^{11} \\ l^{11} & -l^{12} & l^{10} - l^{13} & 0 \\ l^{12} & l^{11} & 0 & l^{10} - l^{13} \end{pmatrix}. \quad (2.160)$$

The matrix of η^1 contains a, b, c, d once in each column, and it is the same for each row in $\eta^{1\dagger}$ and in $-i\eta^{1\dagger}$. There are only two $-$ signs in the right column of η^1. There are two $-$ signs and two $+$ signs in a row and either four $+$ or four $-$ in the other row of $\eta^{1\dagger}$ and of $-i\eta^{1\dagger}$. For the 4×4 matrices, each row and each column contains exactly once the ∂_μ or the $l^{1\mu}$. We count exactly eight $+$ signs and eight $-$ signs in the ∇ matrix, and only four $-$ signs in the p_1 matrix, and these two matrices are symmetric. All this is obviously not at random but emerges from the properties of multiplication in the Cl_3 algebra, which themselves come from the anticommutative property of orthogonal vectors. Thus the Lagrangian density $\mathcal{L}^1$ satisfies:

$$\begin{aligned}0 = \mathcal{L}^1 = &+ a\delta_0 b + c\delta_0 d + (aa + bb + cc + dd)l^{10}\\ &+ b\delta_1 c + d\delta_1 a + (ac + bd + ca + db)l^{11}\\ &+ a\delta_2 c + b\delta_2 d + (ad - bc - cb + da)l^{12}\\ &+ b\delta_3 a + c\delta_3 d = (aa + bb - cc - dd)l^{13}.\end{aligned} \tag{2.161}$$

where we use the notation:

$$u\delta_\mu v := u(\partial_\mu v) - (\partial_\mu u)v. \tag{2.162}$$

We remark that all the differential terms in the Lagrangian density are δ_μ terms. Each variable a, b, c, d is present once and only once with each δ_μ. And similarly each variable a, b, c, d is present once and only once with each p_1^μ. These are all the properties that are necessary and sufficient to allow us to obtain the wave equations from the Lagrange equations, which is what we see now. The Lagrange equation relative to the parameter a is:

$$\frac{\partial \mathcal{L}}{\partial a} = \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu a)}\Big), \tag{2.163}$$

which gives the wave equation:

$$\begin{aligned}&\partial_0 b - \partial_1 d + \partial_2 c - \partial_3 b + 2(al^{10} + cl^{11} + dl^{12} + al^{13})\\ &\qquad = \partial_0(-b) + \partial_1 d + \partial_2(-c) + \partial_3 b.\end{aligned} \tag{2.164}$$

The differential terms of the right part are exactly the opposites of the differential terms of the left part because the Lagrangian density contains only δ_μ. And there is exactly one term of each variable because each variable is contained once and only once with each value of μ. The origin of these properties is indeed the structure of the Cl_3 algebra. The factor of 2 comes from the fact that each variable is present twice as a factor of each $l^{1\mu}$, for the same reasons of structure and signs that result from the anticommutation. As a result we can simplify this wave equation:

$$0 = \partial_0 b - \partial_1 d + \partial_2 c - \partial_3 b + (al^{10} + cl^{11} + dl^{12} + al^{13}). \tag{2.165}$$

We can indeed use the same method to derive each Lagrange equation. For the b variable we obtain:

$$\begin{aligned}&- \partial_0 a + \partial_1 c + \partial_2 d + \partial_3 a + 2(bl^{10} + dl^{11} - cl^{12} + bl^{13})\\ &= \partial_0(a) + \partial_1(-c) + \partial_2(-d) + \partial_3(-a),\\ &0 = -\partial_0 a + \partial_1 c + \partial_2 d + \partial_3 a + (bl^{10} + dl^{11} - cl^{12} + bl^{13}).\end{aligned} \tag{2.166}$$

Next, these two wave equations may be combined into one by introducing $a + ib = \eta_1^1$ and $c + id = \eta_2^1$:

$$0 = (\partial_0 - \partial_3)\eta_1^1 + (-\partial_1 + i\partial_2)\eta_2^1 + i[(l^{10} + l^{13})\eta_1^1 + (l^{11} - il^{12})\eta_2^1]. \tag{2.167}$$

We continue the calculation with the Lagrange equations relative to the c and d parameters. We simplify and group these equations and this gives:

$$0 = (-\partial_1 - i\partial_2)\eta_1^1 + (\partial_0 + \partial_3)\eta_2^1 + i[(l^{11} + il^{12})\eta_1^1 + (l^{10} - l^{13})\eta_2^1]. \quad (2.168)$$

Finally, we combine these two equations into one:

$$\begin{aligned} 0 &= \begin{pmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{pmatrix} \begin{pmatrix} \eta_1^1 \\ \eta_2^1 \end{pmatrix} + i \begin{pmatrix} l^{10} + l^{13} & l^{11} - il^{12} \\ l^{11} + il^{12} & l^{10} - l^{13} \end{pmatrix} \begin{pmatrix} \eta_1^1 \\ \eta_2^1 \end{pmatrix} \\ 0 &= -i\nabla\eta^1 + l^1\eta^1. \end{aligned} \quad (2.169)$$

This calculation is often presented in a very concise manner, using ψ – which is a column matrix with four complex components – as if it could be a real number. This ultra-concise calculation is nevertheless always correct, and our complete and detailed calculation is sufficient proof: the equality $0 = \mathcal{L}$ is the necessary consequence of the wave equations of the four η^1, ξ^1, η^8 and ξ^8. Next this real $\mathcal{L}$ density, as a result of the anticommutation of the basis vectors that is itself a result of the equality $uu = u{\cdot}u$ for any vector u, has as many $-$ signs as $+$ signs. And these signs are distributed such that each real equation is obtained **automatically** as a Lagrange equation from the only equation $0 = \mathcal{L}$ [8]. Moreover, when we vary the density with regard to one of the spinors, we calculate as if the potentials do not depend on these spinors. And indeed we will see that they depend on the spinors as in any true field theory. We will also see that all terms (potentials and mass) present in the Lagrangian density come down *in fine* to linear combinations of the chiral currents D_R^n and D_L^n. It happens that the calculation of the Lagrange equations is nevertheless correct as a result of identities that suppress this dependence. It is the reason why the gauge potentials seem exterior to the wave even if they strictly depend on the spinors. To see this we consider for instance the term: $B_\mu\eta^{1\dagger}\sigma^\mu\eta^1 = \eta^{1\dagger}B\eta^1$. When we derive this term in η^1, we suppose that this derivation does not affect B. Nevertheless this potential B may include a term depending on $\mathrm{D}_L^1 = L^1\widetilde{L}^1$. In practice this term gives no supplementary contribution to the Lagrange equation because

$$\begin{aligned} \eta^{1\dagger}L^1 &= \sqrt{2}\begin{pmatrix} \overline{\eta}_1^1 & \overline{\eta}_2^1 \end{pmatrix} \begin{pmatrix} 0 & -\overline{\eta}_2^1 \\ 0 & \overline{\eta}_1^1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 0 & -\overline{\eta}_1^1\overline{\eta}_2^1 + \overline{\eta}_2^1\overline{\eta}_1^1 \end{pmatrix} = 0. \quad (2.170) \\ \eta^{1\dagger}\mathrm{D}_L^1\eta^1 &= \eta^{1\dagger}L^1\widetilde{L}^1\eta^1 = 0. \end{aligned}$$

8. This double link between the wave equation and Lagrangian density is not a general property of Clifford algebra. There are dimensions and signatures that ensure an automatic derivation of wave equations from the only equation corresponding to the Clifford real part, but other dimensions or signatures do not ensure the same. To explain this it is enough to consider $Cl_{1,4}$, the Clifford algebra of Kaluza's space-time which has a supplementary dimension of space. The reversion and consequently the multiplication on the left side by $\widetilde{\Psi}$ does not satisfy the properties giving the double link between the wave equation and Lagrangian density. Then there is no general principle behind the automatic behaviour of the Lagrangian formalism in relativistic quantum physics: it is only an inevitable consequence of the particular properties of space-time. These special features come from the dimension of time and space that are always respectively one and three in any tangent space to the space-time manifold of GR.

These identities also hide the great similarity between the wave equations of relativistic quantum physics and Einstein's equations of gravitation, which are highly nonlinear.

In Lagrangian physics, the Lagrange equation is obtained by neglecting the term remaining after an integration by parts, which supposes that these terms may be neglected. In fact the thing that is neglected is just the proof. It was never really understood **why** Maxwell's equations governing the electromagnetic field or Einstein's equations governing the gravitational field must necessarily be derived from a Lagrangian formalism. The Lagrange equations are indeed used to obtain a part of laws of electromagnetism, the part that links derivatives of fields to currents (details in A.3.6). This calculation is perfectly correct but laws linking fields to potentials do not come from Lagrange equations. Moreover the propagation of the fields as far from the sources as desired, is not accounted for. This propagation *ad infinitum* without attenuation other than due to the distance from the source could invalidate the cancellation of the terms remaining after integration by parts: the increase of the volume exactly offsets the decrease in the magnitude of the potential terms.

If this Lagrangian physics acts perfectly, if the terms which could not disappear are fortunately suppressed, this does not come from a metaphysical principle but rather results from the particular properties of the Cl_3 algebra. The mechanism has a purely algebraic origin, and is always valid for all interactions of physics, since it is directly linked to the dimension of space and the signature of space-time geometry. The equivalence between the usual form and the completely invariant form of the wave equations implies the obtaining of the real equations forming the system of either linear or improved Dirac equation. This necessary obtaining is called the Lagrange equations.

Therefore we are always correct when we use Noether's theorem to link conservative quantities to the invariance of the Lagrangian density. We note here that it is enough to change the dimension or the signature of space-time to eventually lose the Lagrangian mechanism that is essential for the functioning of matter dynamics. [9]

The detailed study that we carried out on the $\mathcal{L}^1$ part of the Lagrangian density may indeed be extended to the three other parts of the leptonic wave and also to the antileptonic wave, which is the same wave simply with the change of ∇ into $-\nabla$ in the wave equation and the exchange of left and right terms, and thus also in the Lagrangian density that is its real part.

9. This is a sufficient reason for the general failure of theories with a great number of dimensions to obtain a realistic picture of the Standard Model of quantum physics. It is the same for any theory based on Riemannian manifolds, and using only an indeterminate n dimension.

2.3.5 Iteration and equations of gauge fields

For the leptonic wave the Lagrangian density of the Standard Model is made of two parts: a part describing the quantum wave of the electron and its neutrino and another part describing the gauge bosons. This means B and W^j, which are necessary for the gauge invariance. We automatically obtained the lepton part of the Lagrangian density from the equation of the wave with spin 1/2. It is not at all the same for the boson part that is *not* relativistically linked to spinors. We saw in the first chapter that the classical link between potentials and fields is not the true one since the electromagnetic field is a field of operators. The existence of a Lagrangian density for the boson part is questionable. We have nevertheless the possibility of completely avoiding this not-fully-justified part of the Lagrangian density by the use of the fully-justified part and only this part. This part of the Lagrangian density comes from the fermionic wave equations which we will use with a functional recursive form. We again use the decomposition (2.89) of Ψ_l into its four chiral parts. We let:

$$\mathrm{p}_1 := l^1;\ \mathrm{p}_2 := r^1;\ \mathrm{p}_3 := l^8;\ \mathrm{p}_4 := r^8;\ \mathrm{p}_n^{-1} = \mathrm{p}_{n\mu}^{-1}\widehat{\sigma}^\mu. \tag{2.171}$$

With (2.145) to (2.148) we get

$$\eta^1 = i\mathrm{p}_1^{-1}\nabla\eta^1;\ \nabla\eta^1 = -i\mathrm{p}_1\eta_1, \tag{2.172}$$
$$\xi^1 = i\widehat{\mathrm{p}}_2^{-1}\widehat{\nabla}\xi^1;\ \widehat{\nabla}\xi^1 = -i\widehat{\mathrm{p}}_2\xi^1, \tag{2.173}$$

$$\eta^8 = i\mathrm{p}_3^{-1}\widetilde{\nabla}\eta^8;\ \widetilde{\nabla}\eta^8 = -i\mathrm{p}_3\eta^8, \tag{2.174}$$
$$\xi^8 = i\widehat{\mathrm{p}}_4^{-1}\overline{\nabla}\xi^8;\ \overline{\nabla}\xi^8 = -i\widehat{\mathrm{p}}_4\xi^8. \tag{2.175}$$

We then get by iterating these equations:

$$\eta^1 = i\mathrm{p}_1^{-1}\nabla(i\mathrm{p}_1^{-1}\nabla\eta^1) = i\mathrm{p}_1^{-1}\nabla[i\mathrm{p}_1^{-1}\nabla(i\mathrm{p}_1^{-1}\nabla\eta^1)] \tag{2.176}$$
$$\xi^1 = i\widehat{\mathrm{p}}_2^{-1}\widehat{\nabla}(i\widehat{\mathrm{p}}_2^{-1}\widehat{\nabla}\xi^1) = i\widehat{\mathrm{p}}_2^{-1}\widehat{\nabla}[i\widehat{\mathrm{p}}_2^{-1}\widehat{\nabla}(i\widehat{\mathrm{p}}_2^{-1}\widehat{\nabla}\xi^1)], \tag{2.177}$$
$$\eta^8 = i\mathrm{p}_3^{-1}\widetilde{\nabla}(i\mathrm{p}_3^{-1}\widetilde{\nabla}\eta^8) = i\mathrm{p}_3^{-1}\widetilde{\nabla}[i\mathrm{p}_3^{-1}\widetilde{\nabla}(i\mathrm{p}_3^{-1}\widetilde{\nabla}\eta^8)], \tag{2.178}$$
$$\xi^8 = i\widehat{\mathrm{p}}_4^{-1}\overline{\nabla}(i\widehat{\mathrm{p}}_4^{-1}\overline{\nabla}\xi^8) = i\widehat{\mathrm{p}}_4^{-1}\overline{\nabla}[i\widehat{\mathrm{p}}_4^{-1}\overline{\nabla}(i\widehat{\mathrm{p}}_4^{-1}\overline{\nabla}\xi^8)]. \tag{2.179}$$

These equations are not optional; they are an obligatory consequence of the wave equation of each spinor. Wave equations, iterated once, allow us to define gauge fields from potential and mass terms included in the p_n^{-1}. The Standard Model has problems with the Yang–Mills fields, and we can now see one reason for these difficulties: Yang–Mills fields are not independent from the quantum wave since they are defined from wave equations. And there are four kinds of definitions following the four kinds of representations of Cl_3^*. We now replace the column-spinors ξ and η by the corresponding elements in Cl_3. We get:

$$\nabla\widehat{L}^1 = -i\mathrm{p}_1\widehat{L}^1;\ \widehat{\nabla}(\nabla\widehat{L}^1) = -i\widehat{\nabla}(\mathrm{p}_1\widehat{L}^1), \tag{2.180}$$
$$\widehat{\nabla}R^1 = -i\widehat{\mathrm{p}}_2R^1;\ \nabla(\widehat{\nabla}R^1) = -i\nabla(\widehat{\mathrm{p}}_2R^1), \tag{2.181}$$
$$\nabla\overline{L}^8 = -i\mathrm{p}_3\overline{L}^8;\ \widehat{\nabla}(\nabla\overline{L}^8) = -i\widehat{\nabla}(\mathrm{p}_3\overline{L}^8), \tag{2.182}$$
$$\widehat{\nabla}\widetilde{R}^8 = -i\widehat{\mathrm{p}}_4\widetilde{R}^8;\ \nabla(\widehat{\nabla}\widetilde{R}^8) = -i\nabla(\widehat{\mathrm{p}}_4\widetilde{R}^8). \tag{2.183}$$

We now define the gauge fields F as:

$$\widehat{\nabla}(\mathrm{p}_1\widehat{L}^1) = F_L^1(\widehat{L}^1) + \widehat{\mathrm{p}}_1\nabla\widehat{L}^1, \tag{2.184}$$
$$\nabla(\widehat{\mathrm{p}}_2R^1) = \widehat{F}_R^1(R^1) + \mathrm{p}_2\widehat{\nabla}R^1, \tag{2.185}$$
$$\widehat{\nabla}(\mathrm{p}_3\overline{L}^8) = F_L^8(\overline{L}^8) + \widehat{\mathrm{p}}_3\nabla\overline{L}^8, \tag{2.186}$$
$$\nabla(\widehat{\mathrm{p}}_4\widetilde{R}^8) = \widehat{F}_R^8(R^8) + \mathrm{p}_4\widehat{\nabla}\widetilde{R}^8. \tag{2.187}$$

In any physical theory of fields, the link between potential terms and field terms cannot be arbitrary. For instance, the gravitational potential of the sun is not postulated but calculated from the equations of the gravitational field. **The main novelty of the previous relations is that the gauge fields are different with left waves and with right waves**. And we must recall that a photon is always a purely left or purely right wave. Since:

$$\Box\widehat{L}^1 = -i[F_L^1(\widehat{L}^1) + \widehat{\mathrm{p}}_1\nabla\widehat{L}^1] = -iF_L^1(\widehat{L}^1) - i\widehat{\mathrm{p}}_1(-i\mathrm{p}_1\widehat{L}^1), \tag{2.188}$$

and with:

$$\mathrm{p}_n^2 = \mathrm{p}_n\cdot\mathrm{p}_n = \mathrm{p}_n\widehat{\mathrm{p}}_n = \widehat{\mathrm{p}}_n\mathrm{p}_n, \tag{2.189}$$

The second-order wave equations read:

$$0 = (\Box + \mathrm{p}_1^2 + iF_L^1)(\widehat{L}^1), \tag{2.190}$$
$$0 = (\Box + \mathrm{p}_2^2 + i\widehat{F}_R^1)(R^1), \tag{2.191}$$
$$0 = (\Box + \mathrm{p}_3^2 + iF_L^8)(\overline{L}^8), \tag{2.192}$$
$$0 = (\Box + \mathrm{p}_4^2 + i\widehat{F}_R^8)(\widetilde{R}^8). \tag{2.193}$$

We then have:

$$F_L^1(\widehat{L}^1) = i(\Box + \mathrm{p}_1^2)(\widehat{L}^1), \tag{2.194}$$
$$\widehat{F}_R^1(R^1) = i(\Box + \mathrm{p}_2^2)(R^1), \tag{2.195}$$
$$F_L^8(\overline{L}^8) = i(\Box + \mathrm{p}_3^2)(\overline{L}^8), \tag{2.196}$$
$$\widehat{F}_R^8(\widetilde{R}^8) = i(\Box + \mathrm{p}_4^2)(\widetilde{R}^8). \tag{2.197}$$

2.3.6 Weinberg–Salam angle

This parameter of the Standard Model is an angle which measures the mixing between the photon and the other gauge bosons of the $U(1)\times SU(2)$ group. This θ_W angle satisfies:

$$g_1 = \frac{q}{\cos(\theta_W)} \ ; \ \ g_2 = \frac{q}{\sin(\theta_W)} \ ; \ \ q = \frac{e}{\hbar c} \tag{2.198}$$

$$-g_1 B + g_2 W^3 = \sqrt{g_1^2 + g_2^2} Z^0 = \frac{2q}{\sin(2\theta_W)} Z^0 \tag{2.199}$$

$$B = \cos(\theta_W)\mathrm{A} - \sin(\theta_W) Z^0 \ ; \ W^3 = \sin(\theta_W)\mathrm{A} + \cos(\theta_W) Z^0 \tag{2.200}$$

$$B + iW^3 = e^{i\theta_W}(\mathrm{A} + iZ^0) \ ; \ \ \mathrm{A} + iZ^0 = e^{-i\theta_W}(B + iW^3). \tag{2.201}$$

With the equations (2.139) and (2.140), grouping together the three terms W^j, we replace $3W^3$ by W, and in the place of the previous equations we then let:

$$g_1 = \frac{q}{\cos(\theta_W)} \ ; \ \ g_2 = \frac{q}{\sin(\theta_W)} \ ; \ \ q = \frac{e}{\hbar c} \tag{2.202}$$

$$\mathrm{A} = \cos(\theta_W) B + \sin(\theta_W) W; \ Z^0 = -\sin(\theta_W) B + \cos(\theta_W) W, \tag{2.203}$$

$$B = \cos(\theta_W)\mathrm{A} - \sin(\theta_W) Z^0 \ ; \ W = \sin(\theta_W)\mathrm{A} + \cos(\theta_W) Z^0 \tag{2.204}$$

$$B + iW = e^{i\theta_W}(\mathrm{A} + iZ^0) \ ; \ \ \mathrm{A} + iZ^0 = e^{-i\theta_W}(B + iW). \tag{2.205}$$

The system of wave equations of the electron is now expressed as:

$$\begin{aligned} 0 &= (\nabla + i\frac{g_1}{2} B + i\frac{g_2}{2} W + i\mathbf{l}\mathrm{v})\widehat{L}^1 \\ 0 &= (\nabla - ig_1 B - i\mathbf{r}\mathrm{v})\widehat{R}^1. \end{aligned} \tag{2.206}$$

And with the previous definitions this is equivalent to

$$\begin{aligned} 0 &= [\nabla - i(q\mathrm{A} + \mathbf{r}\mathrm{v}) + iqTZ^0]\widehat{R}^1; \ T = \tan(\theta_W), \\ 0 &= [\nabla + i(q\mathrm{A} + \mathbf{l}\mathrm{v}) + i\frac{q}{2}(-T + \frac{1}{T}) Z^0]\widehat{L}^1. \end{aligned} \tag{2.207}$$

Since there is only one way to express the X and Y terms as sum and difference: $X = 1/2(X+Y)+1/2(X-Y)$ and $Y = 1/2(X+Y)-1/2(X-Y)$, we recast this system in the form:

$$\begin{aligned} 0 &= \Big[\nabla - i[q\mathrm{A} + \mathbf{r}\mathrm{v}] - i\frac{q}{4}(\frac{1}{T} - 3T) Z^0 + i\frac{q}{4}(\frac{1}{T} + T) Z^0\Big]\widehat{R}^1, \\ 0 &= \Big[\nabla + i[q\mathrm{A} + \mathbf{l}\mathrm{v}] + i\frac{q}{4}(\frac{1}{T} - 3T) Z^0] + i\frac{q}{4}(\frac{1}{T} + T) Z^0\Big]\widehat{L}^1. \end{aligned} \tag{2.208}$$

We obtain the wave equation of the electron (1.147) only if the Z^0 terms have only one sign, positive, thus only if $3T - 1/T$ is cancelled. And this is just the case if $\theta_W = 30°$, which we obtained in [32] via another reasoning, independent of the previous one. Moreover this result was also obtained by Stoica in a completely different manner [103], which supports this result and gives

$$T = \frac{1}{\sqrt{3}};\ \frac{1}{T} = \sqrt{3} = 3T;\ 3T - \frac{1}{T} = 0;\ \frac{q}{4}\Big(T + \frac{1}{T}\Big) = \frac{q}{\sqrt{3}}. \qquad (2.209)$$

We then have:

$$\begin{aligned} 0 &= \Big[\nabla - i(q\mathrm{A} + \mathbf{r}\mathrm{v}) + i\frac{q}{\sqrt{3}}Z^0\Big]\widehat{R}^1, \\ 0 &= \Big[\nabla + i(q\mathrm{A} + \mathbf{l}\mathrm{v}) + i\frac{q}{\sqrt{3}}Z^0\Big]\widehat{L}^1. \end{aligned} \qquad (2.210)$$

The rotation of 30° that the Weinberg–Salam angle makes, is thus shown to be the simple rewriting of the gauge terms as sum and difference of terms that apply to the left spinor and the right spinor of the electron. Moreover it turns out that the calculation of this angle from the experimental data through the approximation method of quantum field theory gives a value near 30° and which gets closer for the data with low energy–momentum.

2.3.7 Consequence for the neutrino–monopole

The Weinberg–Salam angle links several properties: the Z^0 boson has a proper mass greater than that of W^n bosons. The experimental ratio of masses is in the vicinity of the $2/\sqrt{3}$ ratio resulting from the 30° value of the Weinberg–Salam angle. Other properties are the null electric charge of the neutrino and the null proper mass of the photon. The equations of the left and right waves of the neutrino–monopole are now:

$$0 = (-i\nabla + \mathrm{b} - 3\mathrm{w}^3 + m_l\mathrm{v})\overline{L}^8, \qquad (2.211)$$

$$0 = (i\nabla + 2p\mathrm{b} + m_r\mathrm{v})\overline{R}^8. \qquad (2.212)$$

With the 30° value of the angle, this becomes:

$$0 = \Big[-i\nabla + \frac{q}{2}(A - Z'^0) - \frac{q}{2}(A + Z'^0) + m_l\mathrm{v}\Big]\overline{L}^8, \qquad (2.213)$$

$$0 = [i\nabla + pq(A - Z'^0) + m_r\mathrm{v}]\overline{R}^8;\ Z'^0 := \frac{Z^0}{\sqrt{3}}. \qquad (2.214)$$

The potential A cancels out in the wave equation of $\overline{L}^8$, and this is the reason for the neutrino being neutral, which means without electric interaction. We recall that this system is equivalent to the single equation summing the two equations of the system, because $\overline{L}^8$ and $\overline{R}^8$ are two independent columns of $\overline{\phi}^8$:

$$0 = \nabla\overline{\phi}^8(-i\sigma_3) - 2qZ'^0\overline{L}^8 + pqA\overline{R}^8 - pqZ'^0\overline{R}^8 + m_l\mathrm{v}\overline{L}^8 + m_r\mathrm{v}\overline{R}^8 \quad (2.215)$$

If $p = -2$, a value that we will also explain in Chapter 4, we can put together the two terms containing the Z^0 boson:

$$0 = \nabla\overline{\phi}^8\sigma_{21} + qA\overline{\phi}^8(1-\sigma_3) - 2qZ'^0\overline{\phi}^8\sigma_3 + \mathrm{v}\overline{\phi}^8\begin{pmatrix} m_l & 0 \\ 0 & m_r \end{pmatrix}, \tag{2.216}$$
$$0 = \nabla\overline{\phi}^8\sigma_{21} + qA\overline{\phi}^8(1-\sigma_3) - 2qZ'^0\overline{\phi}^8\sigma_3 + \mathrm{v}\overline{\phi}^8\mathbf{m}^8;\ \mathbf{m}^8 := \begin{pmatrix} m_l & 0 \\ 0 & m_r \end{pmatrix}.$$

2.4 Energy–momentum tensor

The Dirac equation uses a unique Lagrangian density but in fact several different Lagrangian densities are possible, all stationary because identically null. In (2.153) we let

$$0 = \mathcal{L} = \frac{m}{k\mathbf{l}}\mathcal{L}^1 + \frac{m}{k\mathbf{r}}\mathcal{L}^2 + \frac{m}{km_l}\mathcal{L}^3 + \frac{m}{km_r}\mathcal{L}^4.$$

We encountered in Chapter 1 the other density that may be formed from the single electron wave. These two densities are generalized as:

$$0 = \mathcal{L}^+ = \frac{m}{k\mathbf{l}}\mathcal{L}^1 + \frac{m}{k\mathbf{r}}\mathcal{L}^2 + \frac{m}{km_l}\mathcal{L}^3 + \frac{m}{km_r}\mathcal{L}^4 = \mathcal{L}, \tag{2.217}$$
$$0 = \mathcal{L}^- = \frac{m}{k\mathbf{l}}\mathcal{L}^1 - \frac{m}{k\mathbf{r}}\mathcal{L}^2 + \frac{m}{km_l}\mathcal{L}^3 - \frac{m}{km_r}\mathcal{L}^4.$$

The existence of several other Lagrangian densities is obtained by assigning the two minus signs differently. Each of these Lagrangian densities (2.153) is invariant under Cl_3^* and also invariant under translation. To each of these invariances is associated a conservative current (Noether's theorem). The energy–momentum tensor is the tensor associated with invariance under translation. Tetrode's tensor T is the tensor associated with $\mathcal{L}^+$. The tensor associated with $\mathcal{L}^-$ generalizes the non-interpreted tensor V of Costa de Beauregard [10] [51]. They satisfy

$$T^\mu_\lambda = \Re\Big[\Big(\frac{m}{k\mathbf{l}}\eta^{1\dagger}\sigma^\mu \mathrm{d}^1_{L\lambda}\eta^1 + \frac{m}{k\mathbf{r}}\xi^{1\dagger}\widehat{\sigma}^\mu \mathrm{d}^1_{R\lambda}\xi^1 + \frac{m}{km_l}\eta^{8\dagger}\sigma^\mu \mathrm{d}^8_{L\lambda}\eta^8 + \frac{m}{km_r}\xi^{8\dagger}\widehat{\sigma}^\mu \mathrm{d}^8_{R\lambda}\xi^8\Big)\Big], \tag{2.218}$$

10. In this note Costa de Beauregard pointed out that the V_{ij} tensor is non-interpreted, which means it is without equivalent in classical physics. We may see that this tensor is obtained [22] by replacing γ_0 with γ_3 in the definition of Tetrode's tensor. This replacement also changes the J current into the K current and is equivalent to the passing from $\mathcal{L}^+$ into $\mathcal{L}^-$. This induces the astonishing idea that two energy–momentum tensors exist in the wave of the electron. The existence of two Lagrangian densities and of two energy–momentum tensors was first encountered in de Broglie's theory of the photon [55][56].

$$V^\mu_\lambda = \Re\Big[-\Big(\frac{m}{k\mathbf{l}}\eta^{1\dagger}\sigma^\mu \mathrm{d}^1_{L\lambda}\eta^1 - \frac{m}{k\mathbf{r}}\xi^{1\dagger}\widehat{\sigma}^\mu \mathrm{d}^1_{R\lambda}\xi^1 + \frac{m}{km_l}\eta^{8\dagger}\sigma^\mu \mathrm{d}^8_{L\lambda}\eta^8 - \frac{m}{km_r}\xi^{8\dagger}\widehat{\sigma}^\mu \mathrm{d}^8_{R\lambda}\xi^8\Big)\Big]. \tag{2.219}$$

where the operators d_λ are defined as:

$$\mathrm{d}^1_{L\lambda}\eta^1 = (-i\partial_\lambda + l^1_\lambda)\eta^1, \tag{2.220}$$
$$\mathrm{d}^1_{R\lambda}\xi^1 = (-i\partial_\lambda + r^1_\lambda)\xi^1, \tag{2.221}$$
$$\mathrm{d}^8_{L\lambda}\eta^8 = (-i\partial_\lambda + l^8_\lambda)\eta^8, \tag{2.222}$$
$$\mathrm{d}^8_{R\lambda}\xi^8 = (-i\partial_\lambda + r^8_\lambda)\xi^8. \tag{2.223}$$

The energy–momentum tensor T is thus the sum of four tensors, one for each spinor of the leptonic wave:

$$T = \frac{m}{k\mathbf{l}}T^1_L + \frac{m}{k\mathbf{r}}T^1_R + \frac{m}{km_l}T^8_R + \frac{m}{km_r}T^8_L, \tag{2.224}$$
$$T^{1\mu}_{L\lambda} = \Re(\eta^{1\dagger}\sigma^\mu \mathrm{d}^1_{L\lambda}\eta^1). \tag{2.225}$$

We obtain the three other parts simply by replacing η^1 with ξ^1, η^8 and ξ^8, and by the replacement of the σ^μ with $\widehat{\sigma}^\mu$ whenever we replace η by ξ. It is thus enough to study T^1_L and then to apply this procedure to the others. What we carry out here is the generalization of the study in Chapter 1. And so we may then again use the same method of calculation which, with (1.285), gives:

$$\begin{aligned}\partial_\mu T^{1\mu}_L &= \partial_\mu T^{1\mu}_{L\lambda}\sigma^\lambda = \Re\big[\partial_\mu[\eta^{1\dagger}\sigma^\mu(-i\partial_\lambda + l^1_\lambda)\eta^1]\big]\sigma^\lambda \\ &= \Re\big[\partial_\mu[-i\eta^{1\dagger}\sigma^\mu\partial_\lambda\eta^1 + l^1_\lambda \mathrm{D}^{1\mu}_L]\big]\sigma^\lambda.\end{aligned} \tag{2.226}$$

Next we use the wave equation of η^1, and this gives

$$\nabla\eta^1 = -il^1\eta^1;\ \partial_\mu \mathrm{D}^{1\mu}_L = 0, \tag{2.227}$$

$$\begin{aligned}\partial_\mu T^{1\mu}_L &= \Re\big[[-i(\nabla\eta^1)^\dagger\partial_\lambda\eta^1 - i\eta^{1\dagger}\partial_\lambda(\nabla\eta^1) - \mathrm{D}^{1\mu}_L\partial_\mu l^1_\lambda]\big]\sigma^\lambda \\ &= \Re[-i(i\eta^{1\dagger}l^1)\partial_\lambda\eta^1 - i\eta^{1\dagger}\partial_\lambda(-il^1\eta^1) + (\partial_\mu l^1_\lambda)\mathrm{D}^{1\mu}_L]\sigma^\lambda \\ &= (\partial_\mu l^1_\lambda - \partial_\lambda l^1_\mu)\mathrm{D}^{1\mu}_L\sigma^\lambda.\end{aligned} \tag{2.228}$$

Similarly, with the right wave of the electron, we have:

$$\partial_\mu T^{1\mu}_R = (\partial_\mu r^1_\lambda - \partial_\lambda r^1_\mu)\mathrm{D}^{1\mu}_R\sigma^\lambda, \tag{2.229}$$
$$\partial_\mu T^{8\mu}_L = (\partial_\mu l^8_\lambda - \partial_\lambda l^8_\mu)\mathrm{D}^{8\mu}_L\sigma^\lambda, \tag{2.230}$$
$$\partial_\mu T^{8\mu}_R = (\partial_\mu r^8_\lambda - \partial_\lambda r^8_\mu)\mathrm{D}^{8\mu}_R\sigma^\lambda. \tag{2.231}$$

The complete electromagnetic field F with magnetic monopoles is the sum of a purely electric field that we denote as F^e, and of a purely magnetic field that we denote as F^m. They satisfy the following:

$$\partial_\mu \mathrm{A}^\mu = 0;\ \partial_\mu Z^{0\mu} = 0, \tag{2.232}$$

$$F = F^e + F^m;\ F^e = \nabla\widehat{\mathrm{A}} = \vec{E} + i\vec{H};\ \vec{E} = -\partial_0\vec{\mathrm{A}} - \vec{\partial}\mathrm{A}_0;\ \vec{H} = \vec{\partial}\times\vec{\mathrm{A}}$$

$$F^m = \nabla\widehat{iZ^0} = \vec{E}^m + i\vec{H}^m;\ \vec{E}^m = \vec{\partial}\times\vec{Z}^0;\ \vec{H}^m = \partial_0\vec{Z}^0 + \vec{\partial}Z_0^0. \tag{2.233}$$

We thus have:

$$F^e_{\mu\lambda} = \partial_\mu A_\lambda - \partial_\lambda A_\mu;\ iF^m_{\mu\lambda} = \partial_\mu Z'^0_\lambda - \partial_\lambda Z'^0_\mu, \tag{2.234}$$

$$\partial_\mu T^\mu = \frac{m}{k\mathbf{l}}\partial_\mu T_L^{1\,\mu} + \frac{m}{k\mathbf{r}}\partial_\mu T_R^{1\,\mu} + \frac{m}{km_l}\partial_\mu T_L^{8\mu} + \frac{m}{km_r}\partial_\mu T_R^{8\,\mu} = \partial_\mu T^\mu_\lambda \sigma^\lambda,$$

$$\partial_\mu T^\mu_\lambda = \frac{m}{k\mathbf{l}}\partial_\mu T^{1\,\mu}_{L\lambda} + \frac{m}{k\mathbf{r}}\partial_\mu T^{1\,\mu}_{R\lambda} + \frac{m}{km_l}\partial_\mu T^{8\mu}_{L\lambda} + \frac{m}{km_r}\partial_\mu T^{8\,\mu}_{R\lambda}. \tag{2.235}$$

And we have:

$$l^1 = \mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v} = \frac{q}{2}(A - Z'^0) + \frac{q}{2}(A + Z'^0) + \mathbf{l}\mathrm{v}$$

$$= q(A + Z'^0) + \mathbf{l}\mathrm{v}, \tag{2.236}$$

$$r^1 = 2\mathrm{b} + \mathbf{r}\mathrm{v} = q(A - Z'^0) + \mathbf{r}\mathrm{v}, \tag{2.237}$$

$$l^8 = \mathrm{b} - 3\mathrm{w}^3 + m_l\mathrm{v} = \frac{q}{2}(A - Z'^0) - \frac{q}{2}(A + 3Z'^0) + m_l\mathrm{v}$$

$$= -2qZ'^0 + m_l\mathrm{v}, \tag{2.238}$$

$$r^8 = 2p\mathrm{b} + m_r\mathrm{v} = pq(A - Z'^0) + m_r\mathrm{v}. \tag{2.239}$$

With the left wave of the electron we obtain:

$$\partial_\mu T^{1\mu}_{L\lambda} = (\partial_\mu l^1_\lambda - \partial_\lambda l^1_\mu)\mathrm{D}^{1\mu}_L$$

$$= \Big[q(\partial_\mu A_\lambda - \partial_\lambda A_\mu) + q(\partial_\mu Z'^0_{\ \lambda} - \partial_\lambda Z'^0_{\ \mu}) + \mathbf{l}(\partial_\mu \mathrm{v}_\lambda - \partial_\lambda \mathrm{v}_\mu)\Big]\mathrm{D}^{1\mu}_L$$

$$= (qF^e_{\mu\lambda} + iqF^m_{\mu\lambda} + \mathbf{l}G_{\mu\lambda})\mathrm{D}^{1\mu}_L \tag{2.240}$$

$$G_{\mu\lambda} := \partial_\mu \mathrm{v}_\lambda - \partial_\lambda \mathrm{v}_\mu, \tag{2.241}$$

where G is a bivector similar to the electromagnetic field. With the right wave of the electron we obtain:

$$\partial_\mu T^{1\mu}_{R\lambda} = (\partial_\mu r^1_\lambda - \partial_\lambda r^1_\mu)\mathrm{D}^{1\mu}_R$$

$$= [q(\partial_\mu A_\lambda - \partial_\lambda A_\mu) - (\partial_\mu Z'^0_{\ \lambda} - \partial_\lambda Z'^0_{\ \mu}) + \mathbf{r}(\partial_\mu \mathrm{v}_\lambda - \partial_\lambda \mathrm{v}_\mu)]\mathrm{D}^{1\mu}_R$$

$$= (qF^e_{\mu\lambda} - iqF^m_{\mu\lambda} + \mathbf{r}G_{\mu\lambda})\mathrm{D}^{1\mu}_R. \tag{2.242}$$

With the left wave of the neutrino–monopole we obtain:

$$\partial_\mu T^{8\mu}_{L\lambda} = (\partial_\mu l^8_\lambda - \partial_\lambda l^8_\mu)\mathrm{D}^{8\mu}_L$$

$$= [0(\partial_\mu A_\lambda - \partial_\lambda A_\mu) - 2q(\partial_\mu Z'^0_{\ \lambda} - \partial_\lambda Z'^0_{\ \mu}) + m_l(\partial_\mu \mathrm{v}_\lambda - \partial_\lambda \mathrm{v}_\mu)]\mathrm{D}^{8\mu}_L$$

$$= \Big(0F^e_{\mu\lambda} - 2iqF^m_{\mu\lambda} + m_l G_{\mu\lambda}\Big)\mathrm{D}^{1\mu}_L. \tag{2.243}$$

With the right wave of the neutrino–monopole we obtain:

$$\begin{aligned}\partial_\mu T^{8\mu}_{R\lambda} &= (\partial_\mu r^8_\lambda - \partial_\lambda r^8_\mu)\mathrm{D}^{8\mu}_R \\ &= [pq(\partial_\mu A_\lambda - \partial_\lambda A_\mu) - pq(\partial_\mu Z'^0_\lambda - \partial_\lambda Z'^0_\mu) + m_r(\partial_\mu \mathrm{v}_\lambda - \partial_\lambda \mathrm{v}_\mu)]\mathrm{D}^{8\mu}_R \\ &= \Big(pqF^e_{\mu\lambda} - ipqF^m_{\mu\lambda} + m_r G_{\mu\lambda}\Big)\mathrm{D}^{1\mu}_R. \end{aligned} \quad (2.244)$$

Adding, we get:

$$\begin{aligned}\partial_\mu T^\mu_\lambda = \frac{q}{k}F^e_{\mu\lambda}\Big(\frac{m}{\mathbf{l}}\mathrm{D}^{1\mu}_L + \frac{m}{\mathbf{r}}\mathrm{D}^{1\mu}_R + \frac{mp}{m_r}\mathrm{D}^{8\mu}_R\Big) \\ + i\frac{q}{k}F^m_{\mu\lambda}\Big(\frac{m}{\mathbf{l}}\mathrm{D}^{1\mu}_L - \frac{m}{\mathbf{r}}\mathrm{D}^{1\mu}_R - 2\frac{m}{m_l}\mathrm{D}^{8\mu}_L - p\frac{m}{m_r}\mathrm{D}^{8\mu}_R\Big) \\ + \frac{m}{k}G_{\mu\lambda}(\mathrm{D}^{1\mu}_L + \mathrm{D}^{1\mu}_R + \mathrm{D}^{8\mu}_L + \mathrm{D}^{8\mu}_R).\end{aligned} \quad (2.245)$$

This gives:

$$\partial_\mu T^\mu = \Big[qF^e_{\mu\lambda}\Big(\underline{\mathrm{J}}^\mu + \frac{mp}{km_r}\mathrm{D}^{8\mu}_R\Big) \quad (2.246)$$

$$+ iqF^m_{\mu\lambda}\Big(\frac{m}{k\mathbf{l}}\mathrm{D}^{1\mu}_L - \frac{m}{k\mathbf{r}}\mathrm{D}^{1\mu}_R - 2\frac{m}{km_l}\mathrm{D}^{8\mu}_L - p\frac{m}{km_r}\mathrm{D}^{8\mu}_R\Big) + \frac{m}{k}G_{\mu\lambda}\mathrm{J}^\mu_l\Big)\Big]\sigma^\lambda.$$

We see that the $\mathrm{D}^{8\mu}_L$ term is missing in the first line: this results from the neutrality of the left wave of the neutrino which does not see the electric interaction (hence the name "neutrino"). When the electron is alone, when weak interactions are not at play, nor the G field, it remains:

$$\partial_\mu T^\mu = qF^e_{\mu\lambda}\Big(\frac{m}{k\mathbf{l}}\mathrm{D}^{1\mu}_L + \frac{m}{k\mathbf{r}}\mathrm{D}^{1\mu}_R\Big)\sigma^\lambda, \quad (2.247)$$

This gives the Lorentz force (1.305) acting on the electric current $\mathrm{j}_e = e(\frac{m}{k\mathbf{l}}\mathrm{D}^1_R + \frac{m}{k\mathbf{r}}\mathrm{D}^1_L)$ of the electron.

2.4.1 Probability density

The component T^0_0 of the energy–momentum tensor satisfies:

$$\begin{aligned}T^0_0 = \Re\Big[-i\Big(\frac{m}{k\mathbf{l}}\eta^{1\dagger}\mathrm{d}^1_{L0}\eta^1 + \frac{m}{k\mathbf{r}}\xi^{1\dagger}\mathrm{d}^1_{R0}\xi^1 \\ + \frac{m}{km_l}\eta^{8\dagger}\mathrm{d}^8_{L0}\eta^8 + \frac{m}{km_r}\xi^{8\dagger}\mathrm{d}^8_{R0}\xi^8\Big)\Big].\end{aligned} \quad (2.248)$$

For a solution to the wave equation with an energy E that is the same for the whole wave, we have:

$$\begin{aligned}-i\mathrm{d}^1_{R0}\xi^1 = \frac{E}{\hbar c}\xi^1(\vec{\mathrm{x}});\ -i\mathrm{d}^8_{R0}\xi^8 = \frac{E}{\hbar c}\xi^8(\vec{\mathrm{x}}), \\ -i\mathrm{d}^1_{L0}\eta^1 = \frac{E}{\hbar c}\eta^1(\vec{\mathrm{x}});\ -i\mathrm{d}^8_{L0}\eta^8 = \frac{E}{\hbar c}\eta^8(\vec{\mathrm{x}}).\end{aligned} \quad (2.249)$$

We then have:

$$\begin{aligned}T_0^0 &= \frac{E}{\hbar c}\Big(\frac{m}{k\mathbf{l}}\eta^{1\dagger}\eta^1 + \frac{m}{k\mathbf{r}}\xi^{1\dagger}\xi^1 + \frac{m}{km_l}\eta^{8\dagger}\eta^8 + \frac{m}{km_r}\xi^{8\dagger}\xi^8\Big)\\ &= \frac{E}{\hbar c}\Big(\frac{m}{k\mathbf{l}}D_L^{10} + \frac{m}{k\mathbf{r}}D_R^{10} + \frac{m}{km_l}D_L^{80} + \frac{m}{km_r}D_R^{80}\Big) = \frac{E}{\hbar c}\mathrm{J}_l^0, \qquad (2.250)\end{aligned}$$

It is the J_l current which is a generalization of the J current in Chapter 1. The reason of the existence of a probability current in physics is the same: the equivalence between inertial mass and gravitational mass which implies:

$$E = \iiint dv T_0^0 \ ; \quad \iiint dv \frac{\mathrm{J}_l^0}{\hbar c} = 1. \qquad (2.251)$$

2.5 Quantization of the kinetic momentum

Noether's theorem derives the conservation of energy–momentum from the invariance of the Lagrangian density under translation. In the same way this same theorem derives the conservation of the kinetic momentum from the invariance of the Lagrangian density under space-time rotations. Relativistic mechanics replaced the group of spatial rotations with the Lorentz group, but quantum theory also replaced this group with the $SL(2,\mathbb{C})$ group. And we extended this invariance using the greater group $GL(2,\mathbb{C}) = Cl_3^*$. We thus start from the real Lagrangian density $\mathcal{L}^-$ and from the energy–momentum corresponding to this Lagrangian:

$$\begin{aligned}V_\lambda^\mu = \Re\Big[- i\Big(&\frac{m}{k\mathbf{l}}\eta^{1\dagger}\sigma^\mu \mathrm{d}_{L\lambda}^1\eta^1 - \frac{m}{k\mathbf{r}}\xi^{1\dagger}\widehat{\sigma}^\mu \mathrm{d}_{R\lambda}^1\xi^1\\ &+ \frac{m}{km_l}\eta^{8\dagger}\sigma^\mu \mathrm{d}_{L\lambda}^8\eta^8 - \frac{m}{km_r}\xi^{8\dagger}\widehat{\sigma}^\mu \mathrm{d}_{R\lambda}^8\xi^8\Big)\Big]. \qquad (2.252)\end{aligned}$$

We note two possible methods for the demonstration of Noether's theorem. However that of Lasenby [83] in the Clifford algebra does not specify in detail what comes with a wave with spin 1/2. Thus we will take the method usual in quantum field theory and we will follow Bailin [2]. We consider a general transformation of the form:

$$M = 1 + \frac{1}{2}(\delta\omega^0 + \delta\omega^1\sigma_1 + \delta\omega^2\sigma_2 + \delta\omega^3\sigma_3 + \delta\omega^4 i\sigma_1 + \delta\omega^5 i\sigma_2 + \delta\omega^6 i\sigma_3 + \delta\omega^7 i) \qquad (2.253)$$

where the eight $\delta\omega^n$ are infinitely small. We have

$$\begin{aligned}M^\dagger &= 1 + \frac{1}{2}(\delta\omega^0 + \delta\omega^1\sigma_1 + \delta\omega^2\sigma_2 + \delta\omega^3\sigma_3 - \delta\omega^4 i\sigma_1\\ &\qquad - \delta\omega^5 i\sigma_2 - \delta\omega^6 i\sigma_3 - \delta\omega^7 i)\\ \mathrm{x}' &= \mathrm{x}'^\mu\sigma_\mu = M\mathrm{x}M^\dagger = \mathrm{x} + \delta\mathrm{x}^\mu\sigma_\mu;\ \ \delta\mathrm{x}^\mu = X_i^\mu\delta\omega^i \qquad (2.254)\end{aligned}$$

This gives

$$\begin{aligned}\delta \mathrm{x}^0 &= \mathrm{x}^0\delta\omega^0 + \mathrm{x}^1\delta\omega^1 + \mathrm{x}^2\delta\omega^2 + \mathrm{x}^3\delta\omega^3,\\ \delta \mathrm{x}^1 &= \mathrm{x}^0\delta\omega^1 + \mathrm{x}^1\delta\omega^0 + \mathrm{x}^2\delta\omega^6 - \mathrm{x}^3\delta\omega^5,\\ \delta \mathrm{x}^2 &= \mathrm{x}^0\delta\omega^2 - \mathrm{x}^1\delta\omega^6 + \mathrm{x}^2\delta\omega^0 + \mathrm{x}^3\delta\omega^4,\\ \delta \mathrm{x}^3 &= \mathrm{x}^0\delta\omega^3 + \mathrm{x}^1\delta\omega^5 - \mathrm{x}^2\delta\omega^4 + \mathrm{x}^3\delta\omega^0.\end{aligned} \tag{2.255}$$

The only non-null X_i^μ are then

$$\begin{aligned}&X_0^0 = \mathrm{x}^0;\ X_1^0 = \mathrm{x}^1;\ X_2^0 = \mathrm{x}^2;\ X_3^0 = \mathrm{x}^3,\\ &X_0^1 = \mathrm{x}^1;\ X_1^1 = \mathrm{x}^0;\ X_5^1 = -\mathrm{x}^3;\ X_6^1 = \mathrm{x}^2,\\ &X_0^2 = \mathrm{x}^2;\ X_2^2 = \mathrm{x}^0;\ X_6^2 = -\mathrm{x}^1;\ X_4^2 = \mathrm{x}^3,\\ &X_0^3 = \mathrm{x}^3;\ X_3^3 = \mathrm{x}^0;\ X_4^3 = -\mathrm{x}^2;\ X_5^3 = \mathrm{x}^1,\end{aligned} \tag{2.256}$$

Bailin denotes the different fields φ_a, and their variations are denoted as

$$\delta\varphi_a = \phi_i^a\delta\omega^i. \tag{2.257}$$

Since we may use the adjoint to obtain the real part, we can opt to consider only four spinor fields:

$$\varphi_1 = \eta^1;\ \varphi_2 = \xi^1;\ \varphi_3 = \eta^8;\ \varphi_4 = \xi^8. \tag{2.258}$$

And we have

$$\begin{aligned}&\eta^1 + \delta\eta^1 = \widehat{M}\eta^1;\ \xi^1 + \delta\xi^1 = M\xi^1;\ \eta^8 + \delta\eta^8 = \widehat{M}\eta^8;\ \xi^8 + \delta\xi^8 = M\xi^8,\\ &\widehat{M} = 1 + \frac{1}{2}(\delta\omega^0 - \delta\omega^1\sigma_1 - \delta\omega^2\sigma_2 - \delta\omega^3\sigma_3\\ &\qquad + \delta\omega^4 i\sigma_1 + \delta\omega^5 i\sigma_2 + \delta\omega^6 i\sigma_3 - \delta\omega^7 i).\end{aligned} \tag{2.259}$$

This gives

$$\begin{aligned}2\delta\xi^1 &= \delta\omega^0\xi^1 + \delta\omega^1\sigma_1\xi^1 + \delta\omega^2\sigma_2\xi^1 + \delta\omega^3\sigma_3\xi^1\\ &\quad + \delta\omega^4 i\sigma_1\xi^1 + \delta\omega^5 i\sigma_2\xi^1 + \delta\omega^6 i\sigma_3\xi^1 + \delta\omega^7 i\xi^1,\end{aligned} \tag{2.260}$$

$$\begin{aligned}2\delta\eta^1 &= \delta\omega^0\eta^1 - \delta\omega^1\sigma_1\eta^1 - \delta\omega^2\sigma_2\eta^1 - \delta\omega^3\sigma_3\eta^1\\ &\quad + \delta\omega^4 i\sigma_1\eta^1 + \delta\omega^5 i\sigma_2\eta^1 + \delta\omega^6 i\sigma_3\eta^1 - \delta\omega^7 i\eta^1.\end{aligned} \tag{2.261}$$

And we obtain two similar formulas for ξ^8 and η^8. With the numbering in (2.258) we get:

$$\phi_0^1 = \frac{\eta^1}{2};\ \phi_1^1 = -\sigma_1\frac{\eta^1}{2};\ \phi_2^1 = -\sigma_2\frac{\eta^1}{2};\ \phi_3^1 = -\sigma_3\frac{\eta^1}{2},$$
$$\phi_4^1 = i\sigma_1\frac{\eta^1}{2};\ \phi_5^1 = i\sigma_2\frac{\eta^1}{2};\ \phi_6^1 = i\sigma_3\frac{\eta^1}{2};\ \phi_7^1 = -i\frac{\eta^1}{2}, \tag{2.262}$$
$$\phi_0^2 = \frac{\xi^1}{2};\ \phi_1^2 = \sigma_1\frac{\xi^1}{2};\ \phi_2^2 = \sigma_2\frac{\xi^1}{2};\ \phi_3^2 = \sigma_3\frac{\xi^1}{2},$$
$$\phi_4^2 = i\sigma_1\frac{\xi^1}{2};\ \phi_5^2 = i\sigma_2\frac{\xi^1}{2};\ \phi_6^2 = i\sigma_3\frac{\xi^1}{2};\ \phi_7^2 = i\frac{\xi^1}{2}, \tag{2.263}$$
$$\phi_0^3 = \frac{\eta^8}{2};\ \phi_1^3 = -\sigma_1\frac{\eta^8}{2};\ \phi_2^3 = -\sigma_2\frac{\eta^8}{2};\ \phi_3^3 = -\sigma_3\frac{\eta^8}{2},$$
$$\phi_4^3 = i\sigma_1\frac{\eta^8}{2};\ \phi_5^3 = i\sigma_2\frac{\eta^8}{2};\ \phi_6^3 = i\sigma_3\frac{\eta^8}{2};\ \phi_7^3 = -i\frac{\eta^8}{2}, \tag{2.264}$$
$$\phi_0^4 = \frac{\xi^8}{2};\ \phi_1^4 = \sigma_1\frac{\xi^8}{2};\ \phi_2^4 = \sigma_2\frac{\xi^8}{2};\ \phi_3^4 = \sigma_3\frac{\xi^8}{2},$$
$$\phi_4^4 = i\sigma_1\frac{\xi^8}{2};\ \phi_5^4 = i\sigma_2\frac{\xi^8}{2};\ \phi_6^4 = i\sigma_3\frac{\xi^8}{2};\ \phi_7^4 = +i\frac{\xi^8}{2}. \tag{2.265}$$

Since the ξ'^n and η'^n are also solutions of the wave equations, the Lagrangian density always satisfies $0 = \mathcal{L}'^-$; thus Noether's theorem associates to each of the eight parameters ω^i a conservative current:

$$j_i^\mu = \Big(\frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}(\partial_\nu\varphi_a) - \mathcal{L}^-\delta_\nu^\mu\Big)X_i^\nu - \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}\phi_i^a. \tag{2.266}$$

In comparison with this general formula we obtain a simplification because our equations are homogeneous, and this is associated to a Lagrangian density that is exactly null. Thus the currents satisfy:

$$j_i^\mu = \Big(\frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}(\partial_\nu\varphi_a)\Big)X_i^\nu - \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}\phi_i^a. \tag{2.267}$$

With (2.149) to (2.152) the Lagrangian density (2.217) gives:

$$\frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_1)} = \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\eta^1)} = -\frac{im}{2k\mathbf{l}}\eta^{1\dagger}\sigma^\mu;\ \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_2)} = \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\xi^1)} = +\frac{im}{2k\mathbf{r}}\xi^{1\dagger}\sigma^\mu, \tag{2.268}$$
$$\frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_3)} = \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\eta^8)} = -\frac{im}{2km_1}\eta^{8\dagger}\sigma^\mu;\ \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_4)} = \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\xi^8)} = +\frac{im}{2km_2}\xi^{8\dagger}\sigma^\mu.$$

Before, quantum theory only used quantities j_1^μ to j_6^μ. These six space-time vectors join two other vectors, and it is precisely one of these new vectors, j_7, that we will use now. We have

$$j_7^\mu = \Big(\frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}(\partial_\nu\varphi_a)\Big)X_7^\nu - \frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}\phi_7^a. \tag{2.269}$$

The only X_i^ν that are not null are listed in (2.256), and this list contains no X_7^ν. This comes from the commutative property: the generator i of the chiral gauge $U(1)$ belongs to the kernel of the homomorphism $f : M \mapsto R$ from Cl_3^* into the D^* group of similitudes (see 1.1.2 and 1.2). We then have:

$$j_7^\mu = -\frac{\partial \mathcal{L}^-}{\partial(\partial_\mu \varphi_a)}\phi_7^a. \tag{2.270}$$

With equations (2.262) to (2.265) and (2.268), and since the adjoint of a real is likewise real, we get

$$\begin{aligned} j_7^\mu = {}& 2\frac{im}{2k\mathbf{l}}\eta^{1\dagger}\sigma^\mu(-i)\frac{\eta^1}{2} - 2\frac{im}{2k\mathbf{r}}\xi^{1\dagger}\sigma^\mu(+i)\frac{\xi^1}{2} \\ & + 2\frac{im}{2km_l}\eta^{8\dagger}\sigma^\mu(-i)\frac{\eta^8}{2} - 2\frac{im}{2km_r}\xi^{8\dagger}\sigma^\mu(+i)\frac{\xi^8}{2}, \end{aligned} \tag{2.271}$$

which means:

$$j_7 = \frac{1}{2}\Big(\frac{m}{k\mathbf{l}}\mathrm{D}_L^1 + \frac{m}{k\mathbf{r}}\mathrm{D}_R^1 + \frac{m}{km_l}\mathrm{D}_L^8 + \frac{m}{km_r}\mathrm{D}_R^8\Big) = \frac{1}{2}\underline{\mathrm{J}}_l^0. \tag{2.272}$$

With (2.251), the proper kinetic momentum, usually called spin, satisfies :

$$\iiint dv\frac{1}{c}j_7^0 = \frac{1}{2c}\iiint dv\underline{\mathrm{J}}_l^0 = \frac{\hbar}{2}. \tag{2.273}$$

We recall that this equality is obtained from the equivalence principle: the total energy of the electron is equal to the sum over all space of the energy density T_0^0. We may then say that both quantization and general relativity result from the same equivalence principle between inertial mass-energy and gravitational mass-energy. This quantization of the spin from properties of the wave equation was not obtained previously for two reasons: first, nobody suspected the presence of a form invariance group more binding than the Lorentz group [11]. Second reason, nobody except the wise O. Costa de Beauregard [51], saw the existence of the strange V tensor in the wave of the electron.

This quantization of spin concerns the complete lepton, with the two parts that are the electron and the neutrino–monopole. It is the temporal component of a space-time vector which is quantized, and this temporal component is obtained by summing a tensorial quantity over the whole space. **This is both very close to what we know from the experimental point of view, with the true value $\hbar/2$, and very far from current quantum theory since this does not come from the proper value of a Hermitian operator**.

11. This could be seen as early as 1928 but was obscured by the use of infinitesimal transformations which masked the non-equality between $SL(2,\mathbb{C})$ and the Lorentz group.

2.6 Dynamics of the neutrino–monopole

The magnetic monopole may be viewed from three different perspectives because it may be dotted of a right wave, or a left wave or both. The magnetic monopole without a right wave is called a neutrino in today's physics. The $\mathrm{j}_m = g\mathrm{D}_R^8$ is thus null, and afterwards the k_m current is reduced to the left current. The force acting on the neutrino is:

$$\partial_\mu T^\mu = F^m_{\mu\lambda} i\mathrm{k}^\mu_m \sigma^\lambda;\ \mathrm{k}_m = -\frac{2q}{k}\mathrm{D}_L^8, \tag{2.274}$$

$$\partial_\mu T^\mu = \mathrm{f}^0 + \vec{\mathrm{f}};\ \frac{i}{2}F^m_{\mu\lambda}\mathrm{k}^\mu_m\sigma^\lambda = \vec{\mathrm{k}}_m \cdot \vec{H}^m + \mathrm{k}^0_m \vec{H}^m - \vec{\mathrm{k}}_m \times \vec{E}^m, \tag{2.275}$$

$$\frac{\hbar c}{2}\vec{\mathrm{f}} = \mathrm{k}^0_m \vec{H}^m - \vec{\mathrm{k}}_m \times \vec{E}^m;\ F^m = \vec{E}^m + i\vec{H}^m. \tag{2.276}$$

A second possibility which was not yet employed by the Standard Model is that the neutrino–monopole might only have a right wave. In this case it interacts both with particles with electric charge and with particles with magnetic charge, and (2.246) is reduced to

$$\hbar c\partial_\mu T^\mu = F^e_{\mu\lambda}\mathrm{j}^\mu_m\sigma^\lambda + F^m_{\mu\lambda} i\mathrm{k}^\mu_m\sigma^\lambda;\ \mathrm{k}_m = -\frac{pm}{km_r}\mathrm{D}_R^8;\ \mathrm{j}_m = \frac{pm}{km_r}\mathrm{D}_R^8. \tag{2.277}$$

where g is the magnetic charge. We remark that the two currents linked to this right wave are opposite. We must thus expect to get a different result for the force acting on this wave depending on whether an electric charge or a magnetic charge is at play.

The interaction between an electric charge and a magnetic charge (magnetic monopole) was previously described in a complicated way by writing the electromagnetic field of the monopole as if it were of an electric origin: $F = \nabla\widehat{W}$ instead $F = \nabla\widehat{iW}$. Similarly the interaction between a magnetic charge and an electric charge was described by Lochak [84, 85] using the electromagnetic field created by the electron as if it were of a magnetic origin: $F = \nabla\widehat{i\mathrm{A}}$ instead $F = \nabla\widehat{\mathrm{A}}$. But these calculations are correct because we indeed have:

$$F^e_{\mu\lambda}\mathrm{j}^\mu_m = \nabla\widehat{\mathrm{A}}\mathrm{j}^\mu_m = \nabla\widehat{i\mathrm{A}}ig\mathrm{D}^\mu_R. \tag{2.278}$$

We may then refer to these works [84, 85] for the demonstration of Dirac's formula $eg/\hbar c = 1/2$. Since we just explained how the quantization of the kinetic momentum follows from the equivalence principle, since the quantization of the electric charge and of the magnetic charge follows from Dirac's formula, we see how the quantization of charges also follows from the equivalence principle and from the extended invariance.

Chapter 3

Electroweak and strong interactions

We study the subspace of the $Cl_{3,3}$ algebra corresponding to the quark part of the fermionic wave (first generation). In the framework of this algebra we study weak interactions of d and u quarks. We present in the same framework the $SU(3)$ group of chromodynamics. We generalize the mass term of the leptonic wave and we obtain the wave equations of quarks with mass term. These wave equations are form-invariant and gauge-invariant precisely under the gauge group of the Standard Model. The wave equations come from Lagrangian equations, solely derived from algebraic properties of the geometric algebra. The dynamics of the quark wave gives the forces acting on the charged and colored fluid. This implies the quantization of the kinetic momentum of the proton and the neutron as well as the confinement of the quarks. The inclusion of Cl_3^* into $\mathrm{End}(Cl_3)$ fixes the orientation of space. We explain the preference for the left waves.

3.1 The quark sector

We now study the Ψ_q part of the fermionic wave (2.2)

$$\Psi_q := \Psi - \begin{pmatrix} \Psi_l & 0 \\ 0 & \Psi_l \end{pmatrix} = \begin{pmatrix} i\Psi_b & \Psi_r + \Psi_g \\ \Psi_r - \Psi_g & -i\Psi_b \end{pmatrix}, \tag{3.1}$$

$$\Psi_r = \Psi^2 := \begin{pmatrix} -i\phi_{dr} & \phi_{ur}^\dagger \\ \overline{\phi}_{ur} & -i\widehat{\phi}_{dr} \end{pmatrix} = \begin{pmatrix} \phi^2 & \phi^{5\dagger} \\ \overline{\phi}^5 & -\widehat{\phi}^2 \end{pmatrix}, \tag{3.2}$$

$$\Psi_g = \Psi^3 := \begin{pmatrix} -i\phi_{dg} & \phi_{ug}^\dagger \\ \overline{\phi}_{ug} & -i\widehat{\phi}_{dg} \end{pmatrix} = \begin{pmatrix} \phi^3 & \phi^{6\dagger} \\ \overline{\phi}^6 & -\widehat{\phi}^3 \end{pmatrix}, \tag{3.3}$$

$$\Psi_b = \Psi^4 := \begin{pmatrix} -i\phi_{db} & \phi_{ub}^\dagger \\ \overline{\phi}_{ub} & -i\widehat{\phi}_{db} \end{pmatrix} = \begin{pmatrix} \phi^4 & \phi^{7\dagger} \\ \overline{\phi}^7 & -\widehat{\phi}^4 \end{pmatrix}. \tag{3.4}$$

So we replace the index of color r, g, b by an upper numeric index:

$$\Psi_q = \begin{pmatrix} i\Psi^4 & \Psi^2 + \Psi^3 \\ \Psi^2 - \Psi^3 & -i\Psi^4 \end{pmatrix}; \ \Psi^n = \begin{pmatrix} \phi^n & \widetilde{\phi}^{3+n} \\ \overline{\phi}^{3+n} & -\widehat{\phi}^n \end{pmatrix}. \tag{3.5}$$

We use the identity in Cl_3 between the adjoint $\phi^\dagger$ and the reverse $\widetilde{\phi}$. Next the $P : \phi \mapsto \widehat{\phi}$ transformation is the main automorphism in Cl_3 (parity). We may identify Ψ^n with its first row:

$$\Psi^n = \begin{pmatrix} \phi^n & \widetilde{\phi}^{3+n} \\ \overline{\phi}^{3+n} & -\widehat{\phi}^n \end{pmatrix} = \begin{pmatrix} \phi^n & \widetilde{\phi}^{3+n} \end{pmatrix}, \tag{3.6}$$

easing calculations with $Cl_3 \times Cl_3$. The two supplementary dimensions of time that should allow us to pass from $Cl_{1,3}$ to $Cl_{3,3}$ do not have physical reality. This $Cl_{3,3}$ is interesting only because $Cl_{3,3} = \mathrm{End}(Cl_3)$. The six R^n and the six L^n are the only mathematical objects that are really important in this chapter:

$$R^n = \phi^n \frac{1+\sigma_3}{2}; \ L^n = \phi^n \frac{1-\sigma_3}{2}; \ n = 2,3,4, \tag{3.7}$$

$$\widetilde{R}^{3+n} = \widetilde{\phi}^{3+n} \frac{1+\sigma_3}{2}; \ \widetilde{L}^{3+n} = \widetilde{\phi}^{3+n} \frac{1-\sigma_3}{2}. \tag{3.8}$$

As previously, electroweak interactions (and further strong interactions) are obtained by replacing partial derivatives with gauge-invariant derivatives. We always use the notation of B.2. We indicate in which algebra we are calculating as follows: The same vectors of space-time are underlined when we express them in $Cl_{3,3}$. They are in bold when we express them in $Cl_{1,3}$ and will be in ordinary or in Roman when we express them in Cl_3. In this chapter we will use the index 0 for the time component of space-time vectors, indices 4 and 5 being those of the two fictitious supplementary dimensions. We let:

$$\begin{aligned} &\underline{W}^j = \Gamma^\mu W^j_\mu, \ j = 1,2,3 \ ; \ \underline{D} = \Gamma^\mu D_\mu \ ; \ \Gamma^0 = \Gamma_0 \ ; \ \Gamma^j = -\Gamma_j; \ \underline{\mathbf{i}} = \Gamma_{0123}, \\ &\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}; \ \mathbf{W}^j = W^j_\mu \gamma^\mu = \begin{pmatrix} 0 & W^j \\ \widehat{W}^j & 0 \end{pmatrix}; \ W^j = W^j_\mu \sigma^\mu, \\ &\underline{D} = \begin{pmatrix} 0 & \mathbf{D} \\ \mathbf{D} & 0 \end{pmatrix}; \ \mathbf{D} = D_\mu \gamma^\mu = \begin{pmatrix} 0 & D \\ \widehat{D} & 0 \end{pmatrix}; \ D = D_\mu \sigma^\mu. \end{aligned} \tag{3.9}$$

The derivation for the electroweak gauge now reads as:

$$
\begin{aligned}
&\underline{D}(\Psi) = \underline{\partial}(\Psi) + \frac{g_1}{2}\underline{B}\ \underline{P}_0(\Psi) + \frac{g_2}{2}\underline{W}^j\underline{P}_j(\Psi),\\
&\quad\underline{\partial} = \Gamma^\mu\partial_\mu = \begin{pmatrix} 0 & \boldsymbol{\partial} \\ \boldsymbol{\partial} & 0\end{pmatrix};\ \boldsymbol{\partial} = \gamma^\mu\partial_\mu = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0\end{pmatrix};\ \nabla = \sigma^\mu\partial_\mu, \qquad (3.10)\\
&\quad\underline{B} = \Gamma^\mu B_\mu = \begin{pmatrix} 0 & \mathbf{B} \\ \mathbf{B} & 0\end{pmatrix};\ \mathbf{B} = B_\mu\gamma^\mu = \begin{pmatrix} 0 & B \\ \widehat{B} & 0\end{pmatrix};\ B = B_\mu\sigma^\mu.
\end{aligned}
$$

We use two projectors $\underline{P}_\pm$ satisfying

$$\underline{P}_\pm(\Psi_q) = \frac{1}{2}(\Psi_q \pm \underline{\mathbf{i}}\Psi_q\Gamma_{21});\ P_\pm(\Psi^n) = \frac{1}{2}(\Psi^n \pm \mathrm{i}\Psi^n\gamma_{21}), \qquad (3.11)$$

$$P_+(\Psi^n) = \Psi^n_L;\ P_-(\Psi^n) = \Psi^n_R. \qquad (3.12)$$

And we define $P_j(\Psi^n)$, $j = 1,2,3,\ n = 1,2,3,4$ (we recall that $\Psi_l = \Psi^1$):

$$\underline{P}_1(\Psi) = \Gamma_{0123}P_+(\Psi)\Gamma_{35}, \qquad (3.13)$$

$$\underline{P}_2(\Psi) = \Gamma_{0123}P_+(\Psi)\Gamma_{5012}, \qquad (3.14)$$

$$\underline{P}_3(\Psi) = P_+(\Psi)(-\Gamma_{0123}), \qquad (3.15)$$

$$\underline{P}_j(\Psi) = \frac{1}{2}\begin{pmatrix} P_j(\Psi_l) + iP_j(\Psi^4) & P_j(\Psi^2) + P_j(\Psi^3) \\ P_j(\Psi^2) - P_j(\Psi^3) & P_j(\Psi_l) - iP_j(\Psi^4)\end{pmatrix}, j = 0,1,2,3.$$

The three operators $\underline{P}_j$, $j = 1,2,3$ act on the quark sector as they do on the lepton sector:

$$P_1(\Psi^n) = \mathbf{i}P_+(\Psi^n)\gamma_3\gamma_5, \qquad (3.16)$$

$$P_2(\Psi^n) = \mathbf{i}P_+(\Psi^n)(-i\gamma_3), \qquad (3.17)$$

$$P_3(\Psi^n) = P_+(\Psi^n)(-\mathbf{i}). \qquad (3.18)$$

On the contrary the fourth operator acts differently on the wave of leptons and on the quark sector (we will explain this difference at the end of this section). Here we again use the operator P_0 defined in (2.44). The operators acting on the waves of quarks have a similar yet nevertheless different form:

$$
\begin{aligned}
P_0(\Psi_l) &= \Psi_l\gamma_{21} + (1-p)P_-(\Psi_l)\mathbf{i} + p\mathbf{i}P_-(\Psi_l),\\
P_0(\Psi^n) &= -\frac{1}{3}\Psi^n\gamma_{21} + P_-(\Psi^n)\mathbf{i} \qquad (3.19)\\
&= -\frac{1}{3}\Psi^n\gamma_{21} + \frac{1}{2}(\Psi^n\mathbf{i} - \mathbf{i}\Psi^n\gamma_{03}),\ n = 2,3,4.
\end{aligned}
$$

Even if p was null and thus there could not exist any magnetic monopole, an important difference should subsist between (2.44) and (3.19), since the coefficient of $\Psi_l\gamma_{21}$ is 1 while the coefficient of each of the three $\Psi^n\gamma_{21}$ is $-1/3$. We remark that since the quarks have color in triplicates, the sum of the coefficient is $1+3(-1/3)=0$, which indeed is not at random [1]. Next for $n=1,2,3,4$ we let:

$$\frac{\phi^n}{\sqrt{2}} = \begin{pmatrix} \xi_1^n & -\overline{\eta}_2^n \\ \xi_2^n & \overline{\eta}_1^n \end{pmatrix} ; \ \frac{R^n}{\sqrt{2}} = \begin{pmatrix} \xi_1^n & 0 \\ \xi_2^n & 0 \end{pmatrix} ; \ \xi^n = \begin{pmatrix} \xi_1^n \\ \xi_2^n \end{pmatrix} ; \ \widehat{\eta}^n = \begin{pmatrix} -\overline{\eta}_2^n \\ \overline{\eta}_1^n \end{pmatrix} , \quad (3.20)$$

while for $n=5,6,7,8$:

$$\frac{\widetilde{\phi}^n}{\sqrt{2}} = \begin{pmatrix} \xi_1^n & -\overline{\eta}_2^n \\ \xi_2^n & \overline{\eta}_1^n \end{pmatrix} ; \ \frac{\widetilde{R}^n}{\sqrt{2}} = \begin{pmatrix} \xi_1^n & 0 \\ \xi_2^n & 0 \end{pmatrix} ; \ \xi^n = \begin{pmatrix} \xi_1^n \\ \xi_2^n \end{pmatrix} ; \ \widehat{\eta}^n = \begin{pmatrix} -\overline{\eta}_2^n \\ \overline{\eta}_1^n \end{pmatrix} . \quad (3.21)$$

We then get for $n=1,2,3,4$:

$$\frac{\widehat{\phi}^n}{\sqrt{2}} = \begin{pmatrix} \eta_1^n & -\overline{\xi}_2^n \\ \eta_2^n & \overline{\xi}_1^n \end{pmatrix} ; \ \frac{\widehat{L}^n}{\sqrt{2}} = \begin{pmatrix} \eta_1^n & 0 \\ \eta_2^n & 0 \end{pmatrix} ; \ \eta^n = \begin{pmatrix} \eta_1^n \\ \eta_2^n \end{pmatrix} ; \ \widehat{\xi}^n = \begin{pmatrix} -\overline{\xi}_2^n \\ \overline{\xi}_1^n \end{pmatrix} , \quad (3.22)$$

and for $n=5,6,7,8$:

$$\frac{\overline{\phi}^n}{\sqrt{2}} = \begin{pmatrix} \eta_1^n & -\overline{\xi}_2^n \\ \eta_2^n & \overline{\xi}_1^n \end{pmatrix} ; \ \frac{\overline{L}^n}{\sqrt{2}} = \begin{pmatrix} \eta_1^n & 0 \\ \eta_2^n & 0 \end{pmatrix} ; \ \eta^n = \begin{pmatrix} \eta_1^n \\ \eta_2^n \end{pmatrix} ; \ \widehat{\xi}^n = \begin{pmatrix} -\overline{\xi}_2^n \\ \overline{\xi}_1^n \end{pmatrix} . \quad (3.23)$$

P_+ is the projector on left waves and P_- on right waves. For $n=2,3,4$ we have:

$$P_-(\Psi^n) = \begin{pmatrix} R^n & \widetilde{R}^{3+n} \\ \overline{R}^{3+n} & -\widehat{R}^n \end{pmatrix} ; P_+(\Psi^n) = \begin{pmatrix} L^n & \widetilde{L}^{3+n} \\ \overline{L}^{3+n} & -\widehat{L}^n \end{pmatrix} , \quad (3.24)$$

where we recall that the waves numbered $2,3,4$ are the states of color r, g, b of the d quark, while the waves numbered $5,6,7$ are the states of color r, g, b of the u quark. We have the same for the lepton part of the wave with the upper indices 1 and 8 instead of n and $3+n$. We then get for $n=2,3,4$:

$$\begin{aligned} P_0(\Psi^n) &= -\frac{1}{3}\Psi^n\gamma_{21} + P_-(\Psi^n)\mathbf{i} \\ &= \frac{i}{3}\begin{pmatrix} 2R^n + L^n & -4\widetilde{R}^{3+n} + \widetilde{L}^{3+n} \\ 4\overline{R}^{3+n} - \overline{L}^{3+n} & 2\widehat{R}^n + \widehat{L}^n \end{pmatrix} , \quad (3.25) \\ \frac{g_1}{2}\mathbf{B}P_0(\Psi^n) = \mathbf{b}P_0(\Psi^n) &= \frac{i}{3}\begin{pmatrix} \mathrm{b}(4\overline{R}^{3+n} - \overline{L}^{3+n}) & \mathrm{b}(2\widehat{R}^n + \widehat{L}^n) \end{pmatrix} . \end{aligned}$$

Since P_1, P_2 and P_3 remain unchanged when we move on to the quark sector, on the model of (2.57) and (2.59) we have:

$$P_1(\Psi^n) = i\begin{pmatrix} \widetilde{L}^{3+n} & L^n \\ -\widehat{L}^n & \overline{L}^{3+n} \end{pmatrix} ; \ P_2(\Psi^n) = i^2\begin{pmatrix} -\widetilde{L}^{3+n} & L^n \\ \widehat{L}^n & \overline{L}^{3+n} \end{pmatrix} . \quad (3.26)$$

1. This cancellation is very useful in the Standard Model to suppress the "anomalies" linked to the chiral behavior of weak interactions. This played an important role in the discovery of quarks and their three color charges.

For $j = 3$ we get:

$$P_3(\Psi^n) = i \begin{pmatrix} -L^n & \widetilde{L}^{3+n} \\ -\overline{L}^{3+n} & -\widehat{L}^n \end{pmatrix}. \tag{3.27}$$

We then get:

$$\mathbf{w}^j P_j(\Psi^n) = \begin{pmatrix} -i[(\mathbf{w}^1 - i\mathbf{w}^2)\widehat{L}^n + \mathbf{w}^3\overline{L}^{3+n}] & i[(\mathbf{w}^1 + i\mathbf{w}^2)\overline{L}^{3+n} - \mathbf{w}^3\widehat{L}^n] \end{pmatrix}. \tag{3.28}$$

Now (3.10) yields:

$$\begin{aligned} \mathbf{D}\Psi^n &= \boldsymbol{\partial}\Psi^n + \frac{g_1}{2}\mathbf{B}P_0(\Psi^n) + \frac{g_2}{2}\mathbf{W}^j P_j(\Psi^n) \\ &= \boldsymbol{\partial}\Psi^n + \mathbf{b}P_0(\Psi^n) + \mathbf{w}^j P_j(\Psi^n). \end{aligned} \tag{3.29}$$

This gives for the right waves

$$D\widehat{R}^n = \nabla\widehat{R}^n - \frac{2i}{3}\mathrm{b}\widehat{R}^n;\ D\overline{R}^{3+n} = \nabla\overline{R}^{3+n} + \frac{4i}{3}\mathrm{b}\overline{R}^{3+n}. \tag{3.30}$$

And for the left waves we get:

$$D\widehat{L}^n = \nabla\widehat{L}^n - \frac{i}{3}\mathrm{b}\widehat{L}^n - i[(\mathrm{w}^1 + i\mathrm{w}^2)\overline{L}^{3+n} - \mathrm{w}^3\widehat{L}^n], \tag{3.31}$$

$$D\overline{L}^{3+n} = \nabla\overline{L}^{3+n} - \frac{i}{3}\mathrm{b}\overline{L}^{3+n} - i[(\mathrm{w}^1 - i\mathrm{w}^2)\widehat{L}^n + \mathrm{w}^3\overline{L}^{3+n}]. \tag{3.32}$$

Since the operators P_1, P_2 and P_3 act exactly in the same way in the sector of leptons and in the sector of quarks, the gauge invariance that we studied in 2.3 works similarly. This allows us to obtain the gauge field's values. And instead of (2.116) and (2.137) we have:

$$\begin{aligned} &\mathrm{D}_L^{n,\,3+n} - id_L^{n,\,3+n} = 2L^nL^{3+n};\ \mathrm{D}_L^n = L^n\widetilde{L}^n;\ \mathrm{D}_L^{3+n} = \widetilde{L}^{3+n}L^{3+n}, \\ &W_n^1 = \mathrm{D}_L^{n,\,3+n};\ W_n^2 = d_L^{n,\,3+n};\ W_n^3 = \mathrm{D}_L^{3+n} - \mathrm{D}_L^n. \end{aligned} \tag{3.33}$$

We add an index n to the W^j: even if they have the same properties, the W_n^j change with the color or when we pass from leptons to quarks. The gauge invariance is similar to that of the lepton wave. The result, as with the lepton part, is a simplification of the covariant derivatives which become:

$$\begin{aligned} D\widehat{R}^n &= \nabla\widehat{R}^n - \frac{2i}{3}\mathrm{b}\widehat{R}^n, \\ D\widehat{L}^n &= \nabla\widehat{L}^n - \frac{i}{3}\mathrm{b}\widehat{L}^n + 3i\mathrm{w}_3^3\widehat{L}^n;\ \mathrm{w}_n^j = \frac{g_2}{2}W_n^j, \end{aligned} \tag{3.34}$$

$$\begin{aligned} D\overline{R}^{3+n} &= \nabla\overline{R}^{3+n} + \frac{4i}{3}\mathrm{b}\overline{R}^{3+n}, \\ D\overline{L}^{3+n} &= \nabla\overline{L}^{3+n} - \frac{i}{3}\mathrm{b}\overline{L}^{3+n} - 3i\mathrm{w}_n^3\overline{L}^{3+n}. \end{aligned} \tag{3.35}$$

By using a Weinberg–Salam angle of 30° in the lepton case we have:

$$\mathrm{b} = \frac{q}{2}\mathrm{A} - \frac{q}{2\sqrt{3}}Z_n^0;\ 3\mathrm{w}_n^3 = \frac{q}{2}\mathrm{A} + \frac{q\sqrt{3}}{2}Z_n^0. \tag{3.36}$$

We thus have for the d quark:

$$\begin{aligned}
D\widehat{R}^n &= (\nabla - i\frac{q}{3}\mathrm{A} + i\frac{q}{3\sqrt{3}}Z_n^0)\widehat{R}^n,\\
D\widehat{L}^n &= (\nabla + i\frac{q}{3}\mathrm{A} + i\frac{5q}{3\sqrt{3}}Z_n^0)\widehat{L}^n,\\
D\widehat{\phi}^n &= \nabla\widehat{\phi}^n + \frac{q}{3}\mathrm{A}\widehat{\phi}^n\sigma_{12} + i\frac{q}{3\sqrt{3}}Z_n^0(\widehat{R}^n + 5\widehat{L}^n).
\end{aligned} \tag{3.37}$$

This matches well what we expect: The electric charge of the d quark is exactly a third of the charge of the electron (negative). For the u quark we have:

$$\begin{aligned}
D\widehat{R}^{3+n} &= (\nabla + i\frac{2q}{3}\mathrm{A} - i\frac{2q}{3\sqrt{3}}Z_n^0)\widehat{R}^{3+n}\\
D\widehat{L}^{3+n} &= (\nabla - i\frac{2q}{3}\mathrm{A} - i\frac{4q}{3\sqrt{3}}Z_n^0)\widehat{L}^{3+n},\\
D\widehat{\phi}^{3+n} &= \nabla\widehat{\phi}^{3+n} - \frac{2q}{3}\mathrm{A}\widehat{\phi}^{3+n}\sigma_{12} - i\frac{q}{3\sqrt{3}}Z_n^0(2\widehat{R}^{3+n} + 4\widehat{L}^{3+n}).
\end{aligned} \tag{3.38}$$

Here we also obtain the expected result since the charge of the u quark is positive and equal to -2 times the charge of the d quark. What we obtained in the first chapter for the charge conjugation is indeed conserved: the antiquark of d seems to have a charge equal to half of that of the u quark, and the antiquark of u seems to have a charge double that of the d quark. We also recall that the charge conjugation is not only an apparent change of sign of the electric charges: the right and left waves are exchanged. **Here we have an important result which reduces the number of free parameters in the Standard Model**: the simple replacement of the coefficient 1 of $\Psi\gamma_{21}$ by $-1/3$ in $\underline{P}_0$ is enough to obtain the two values of the electric charge of the two kinds of quarks. Thus we have only one free parameter instead two.

3.2 Chromodynamics

The Standard Model considers strong interactions as resulting also from a gauge invariance under a $SU(3)$ color group, from whence comes the word "chromodynamics." We transpose this group to Clifford algebra in a manner similar to that used for weak interactions. We now define Γ_k in a manner similar to $\underline{P}_j$ of the previous section. We know the $i\lambda_k$ generators of the

$SU(3)$ group of chromodynamics:

$$\begin{aligned}
&i\lambda_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; i\lambda_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; i\lambda_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&i\lambda_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \; i\lambda_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; i\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \\
&i\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; i\lambda_8 = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{3.39}$$

Simplifying the notations, we use l, r, g, b instead of Ψ_l, $\Psi_r = \Psi^2$, $\Psi_g = \Psi^3$, $\Psi_b = \Psi^4$. So we have

$$\Psi = \begin{pmatrix} l+ib & r+g \\ r-g & l-ib \end{pmatrix}. \tag{3.40}$$

The unique i of nonrelativistic quantum mechanics must not be confused with the i of the above relation, which is responsible for the orientation of Cl_3. Thus in $Cl_3 \times Cl_3$ we must instead use $\mathbf{i} = \gamma_{0123}$, which does not commute. Therefore (3.39) gives:

$$\begin{aligned}
&i\lambda_1 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} \mathbf{i}g \\ \mathbf{i}r \\ 0 \end{pmatrix}, \; i\lambda_2 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} g \\ -r \\ 0 \end{pmatrix}, \; i\lambda_3 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} \mathbf{i}r \\ -\mathbf{i}g \\ 0 \end{pmatrix}, \\
&i\lambda_4 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} \mathbf{i}b \\ 0 \\ \mathbf{i}r \end{pmatrix}, \; i\lambda_5 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ -r \end{pmatrix}, \; i\lambda_6 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{i}b \\ \mathbf{i}g \end{pmatrix}, \\
&i\lambda_7 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ -g \end{pmatrix}, \; i\lambda_8 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} \mathbf{i}r \\ \mathbf{i}g \\ -2\mathbf{i}b \end{pmatrix}.
\end{aligned} \tag{3.41}$$

The Λ_k corresponding to the $i\lambda_k$ acting on Ψ are:

$$\Lambda_1(\Psi) = -\frac{1}{2}(\Gamma_{45}\Psi + \Gamma_{0123}\Psi S); \; S = \Gamma_{012345}, \tag{3.42}$$

$$\Lambda_2(\Psi) = -\frac{1}{2}(\Gamma_4\Psi\Gamma_{01235} + \Gamma_{01235}\Psi\Gamma_4), \tag{3.43}$$

$$\Lambda_3(\Psi) = \frac{1}{2}(\Gamma_5\Psi\Gamma_{01235} - \Gamma_{01234}\Psi\Gamma_4), \tag{3.44}$$

$$\Lambda_4(\Psi) = \frac{1}{2}(\Gamma_{0123}\Psi\Gamma_4 - \Gamma_{01234}\Psi), \tag{3.45}$$

$$\Lambda_5(\Psi) = -\frac{1}{2}(S\Psi\Gamma_{01235} + \Gamma_{01235}\Psi S) \tag{3.46}$$

$$\Lambda_6(\Psi) = \frac{1}{2}(\Gamma_{01234}\Psi S - \Gamma_{45}\Psi\Gamma_4), \tag{3.47}$$

$$\Lambda_7(\Psi) = \frac{1}{2}(\Gamma_{01235}\Psi - \Psi\Gamma_{01235}), \tag{3.48}$$

$$\Lambda_8(\Psi) = -\frac{1}{2\sqrt{3}}(2\Gamma_{45}\Psi S + \Gamma_5\Psi\Gamma_{01235} + \Gamma_{01234}\Psi\Gamma_4). \tag{3.49}$$

All Λ_k project the Ψ wave onto the quark sector Ψ_q. Thus the lepton part of the wave does not see color forces.

We extend to strong interactions the gauge-invariant derivative of the electroweak interactions (3.10) by letting:

$$\underline{D}(\Psi) = \underline{\partial}(\Psi) + \frac{g_1}{2}\underline{B}\ \underline{P}_0(\Psi) + \frac{g_2}{2}\underline{W}^j\underline{P}_j(\Psi) + \frac{g_3}{2}\underline{G}^k\Lambda_k(\Psi), \tag{3.50}$$

where g_3 is another constant and the $\underline{G}^k$ are eight potential vectors called gluons. Since I_4 commutes with any element in $Cl_{1,3}$ and since $P_j(\mathbf{i}\Psi_{ind}) = \mathbf{i}P_j(\Psi_{ind})$ for $j = 0,1,2,3$ and $ind = l, r, g, b$, we find that each operator $\underline{\mathbf{i}}\Gamma_k$ commutes with all operators $\underline{P}_j$. Now we use twelve reals: a^0, a^j, $j = 1,2,3$, b^k, $k = 1,2,...,8$, and we let:

$$S_0 = a^0\underline{P}_0;\ S_1 = \sum_{j=1}^{j=3} a^j\underline{P}_j;\ S_2 = \sum_{k=1}^{k=8} b^k\Lambda_k;\ \Sigma = S_0 + S_1 + S_2, \tag{3.51}$$

and by using the exponential function we get:

$$\begin{aligned}\exp(\Sigma) &= \exp(S_0)\exp(S_1)\exp(S_2) = \exp(S_1)\exp(S_0)\exp(S_2)\\ &= \exp(S_0)\exp(S_2)\exp(S_1) = \dots\end{aligned} \tag{3.52}$$

in any order, thanks to the commutation of $\underline{P}_0$ with $\underline{P}_j$, $j = 1,2,3$ as well as the commutation of $\underline{P}_j$, $j = 0,1,2,3$ with Λ_k, $k = 1,\dots,8$. The set of $\exp(S)$ operators is a $U(1) \times SU(2) \times SU(3)$ Lie group. The only difference from the Standard Model is: **we do not need to postulate this structure since it results from the calculation of commutators**. The invariance under Cl_3^* (and consequently the relativistic invariance) of this gauge-invariant derivation is similar to that obtained in 2.2.3. The gauge invariance [36][50][45] can be expressed as:

$$\Psi' = [\exp(a^0\underline{P}_0 + S_1 + S_2)](\Psi)\ ;\ \underline{D} = \Lambda^\mu\underline{D}_\mu\ ;\ \underline{D}' = \Lambda^\mu\underline{D}'_\mu, \tag{3.53}$$

$$\underline{D}'_\mu\Psi' = \exp(a^0\underline{P}_0 + S_1 + S_2)\underline{D}_\mu\Psi, \tag{3.54}$$

$$B'_\mu = B_\mu - \frac{2}{g_1}\partial_\mu a^0, \tag{3.55}$$

$$W'^j_\mu \underline{P}_j = \Big[\exp(S_1) W^j_\mu \underline{P}_j - \frac{2}{g_2}\partial_\mu[\exp(S_1)]\Big]\exp(-S_1), \tag{3.56}$$

$$\underline{G}'^k_\mu \Lambda_k = \Big[\exp(S_2) \underline{G}^k_\mu \Lambda_k - \frac{2}{g_3}\partial_\mu[\exp(S_2)]\Big]\exp(-S_2). \tag{3.57}$$

The $SU(3)$ group of chromodynamics generated by the Λ_k operators only acts on the quark sector. By letting:

$$\text{diag}(\Psi) = \frac{1}{4}(\Psi + S\Psi S + \Gamma_4\Psi\Gamma_4 - \Gamma_{01235}\Psi\Gamma_{01235}) = \begin{pmatrix}\Psi_l & 0\\ 0 & \Psi_l\end{pmatrix}, \tag{3.58}$$

we have:

$$\text{diag}\Big([\exp(b^k\Lambda_k)](\Psi)\Big) = \text{diag}(\Psi). \tag{3.59}$$

This comes from the fact that we begin with operators that do not at all act on Ψ_l. For the contrary case to be possible it would be necessary to consider some operators similar to Λ_k coupling the wave of Ψ_l with one of the three Ψ^n waves. This cannot exist because these operators project the right waves onto right waves and the left waves onto left waves, and because the right waves and the left waves of the lepton part, in weak interactions, transform differently compared to the waves of the colored part of the whole wave. We then get a $U(1) \times SU(2) \times SU(3)$ gauge group for a wave incorporating all the fermions of the first generation [2]. This is certainly well established experimentally. The novelty here is simply that this emerges directly from the structure of the quantum wave. Since it is independent of the scale of energies we can understand why the grand unified theories (GUTs) had no success: it is impossible to have a greater group. Thus it is impossible that a quark may be transformed into a lepton. This implies the conservation of a quantity that QFT calls the baryonic number. Moreover this conservation was experimentally supported by neutrino observatories like Kamioka. We may say that **our transposition of the Standard Model into Clifford algebra automatically satisfies this law of conservation**. This is a reinforcement of the Standard Model by concordance with experiment.

3.2.1 Three generations, four neutrinos

The purpose of physical theory is the understanding of experimental facts. Nowadays we must both justify why there are only three kinds of leptons and quarks, and also why there is a fourth neutrino, very different from the three others. Experiments show the existence of only three kinds of light leptons from studying the disintegration of Z^0, and experiments also

2. Later we will see how this group acts on the lepton sector only via the $U(1) \times SU(2)$ part. The physical interpretation is: the leptons are incapable of strong interactions. They interact only via electromagnetic and weak interactions.

suggest the possibility of the existence of a fourth neutrino. We justified the existence of three kinds of leptons in the previous chapter. This is easily generalized to the three generations of the Standard Model. The two other generations are obtained by replacing the σ_3 of the Dirac equation by σ_1 or σ_2 everywhere this direction is in use. Moreover, the passing from one generation to the other must be seen as a circular permutation of the indices $1 \mapsto 2 \mapsto 3 \mapsto 1$ or $1 \mapsto 3 \mapsto 2 \mapsto 1$ for the other generation. For instance, the σ_3 used for the projector defining the right wave and the left wave must be replaced by σ_1 or σ_2. And the σ_1 that links the wave of the particle and the antiparticle must be replaced by σ_2 or σ_3. These changes force us to treat each generation separately, and this explains the separate treatment of each generation in the Standard Model. Yet for a fourth generation a similar case is not possible, because the Cl_3 algebra is the algebra of ordinary 3-dimensional space. There it is impossible to get a fourth set of operators similar to the P_μ.

But the existence of a fourth neutrino [27] is possible because Cl_3 contains four independent terms with square -1. The wave equation of the electron uses one of these four terms: $i\sigma_3 = \sigma_{12}$. Further, the equalities $i\sigma_1 = \sigma_{23}$ and $i\sigma_2 = \sigma_{31}$ explain why two other kinds of leptons exist. We can also build a form-invariant wave equation with the fourth generator $i = \sigma_{123}$:

$$\overline{\phi}(\nabla\widehat{\phi})\sigma_{123} + m\rho = 0. \tag{3.60}$$

Multiplying on the left side by $\overline{\phi}^{\,-1}$ we obtain using $\rho = e^{-i\beta}\overline{\phi}\phi$ the equivalent equation:

$$\nabla\widehat{\phi}i + me^{-i\beta}\phi = 0\ ;\quad \nabla\widehat{\phi} = ime^{-i\beta}\phi. \tag{3.61}$$

We may extend the gauge invariance to a local one:

$$0 = \nabla\widehat{\phi}i + g_1 B\widehat{\phi} + me^{-i\beta}\phi. \tag{3.62}$$

This is equivalent to

$$0 = i\nabla\eta + g_1 B\eta + me^{-i\beta}\xi, \tag{3.63}$$

$$0 = i\nabla\widehat{\xi} + g_1 B\widehat{\xi} + me^{-i\beta}\widehat{\eta}. \tag{3.64}$$

Contrary to our improved wave equation for the electron which has the Dirac equation as its linear approximation, this wave equation cannot come from the linear quantum theory: no linear approximation exists because the angle β is no longer small. This angle is now the phase of the wave. We nevertheless may obtain the plane waves. We search for solutions satisfying

$$\phi = e^{-i\varphi}\phi_0\ ;\quad \varphi = mv_\mu x^\mu\ ;\quad v = \sigma^\mu v_\mu, \tag{3.65}$$

where v is a fixed reduced velocity and ϕ_0 is also a fixed term. We get

$$\nabla\widehat{\phi} = \sigma^\mu\partial_\mu(e^{i\varphi}\widehat{\phi}_0) = imve^{i\varphi}\widehat{\phi}_0. \tag{3.66}$$

And we have

$$\phi\overline{\phi} = e^{-i\varphi}\phi_0 e^{-i\varphi}\overline{\phi}_0 = e^{-2i\varphi}\phi_0\overline{\phi}_0. \tag{3.67}$$

Then if we let

$$\phi_0\overline{\phi}_0 = \rho_0 e^{i\beta_0}, \tag{3.68}$$

we end up with:

$$\beta = \beta_0 - 2\varphi\ ;\ e^{-i\beta}\phi = e^{-i(\beta_0-2\varphi)}e^{-i\varphi}\phi_0 = e^{-i(\beta_0-\varphi)}\phi_0. \tag{3.69}$$

Hence (3.66) is equivalent to

$$imve^{i\varphi}\widehat{\phi}_0 = ime^{-i(\beta_0-\varphi)}\phi_0 \tag{3.70}$$

$$v\widehat{\phi}_0 = e^{-i\beta_0}\phi_0;\ e^{i\beta_0}v\widehat{\phi}_0 = \phi_0. \tag{3.71}$$

By conjugating we get

$$e^{-i\beta_0}\widehat{v}\phi_0 = \widehat{\phi}_0. \tag{3.72}$$

Therefore we have

$$\phi_0 = e^{i\beta_0}v\widehat{\phi}_0 = e^{i\beta_0}v[e^{-i\beta_0}\widehat{v}\phi_0] = v\widehat{v}\phi_0. \tag{3.73}$$

Then if $\phi_0 \neq 0$ we get:

$$1 = v\widehat{v}, \tag{3.74}$$

which gives $v^0 = \sqrt{1+\vec{v}^2}$ or $v^0 = -\sqrt{1+\vec{v}^2}$ and since (3.71) implies:

$$\begin{aligned}&\phi_0 = ve^{i\beta_0}\widehat{\phi}_0;\ \mathrm{D}_0 = \phi_0\widetilde{\phi}_0 = ve^{i\beta_0}\widehat{\phi}_0\widetilde{\phi}_0 = ve^{i\beta_0}\rho_0 e^{-i\beta_0} = v\rho_0,\\ &\mathrm{D}_0^0 = v^0\rho_0;\ v^0 > 0;\ v^0 = \sqrt{1+\vec{v}^2}.\end{aligned} \tag{3.75}$$

Therefore no plane waves can exist with a sign of energy opposite to the sign of the mass. This wave equation may have a gauge term and may be expressed in an invariant manner of the form:

$$0 = \overline{\phi}(\nabla\widehat{\phi})i + \overline{\phi}qB\widehat{\phi} + m\rho. \tag{3.76}$$

Using the reversion we get

$$0 = -i(\overline{\phi}\nabla)\widehat{\phi} + \overline{\phi}qB\widehat{\phi} + m\rho. \tag{3.77}$$

Given (1.120) to (1.123) we have:

$$\overline{\phi}(\nabla\widehat{\phi}) = \frac{1}{2}(\nabla\cdot\mathrm{D}_\mu)\sigma^\mu + iw_\mu\sigma^\mu, \tag{3.78}$$

$$\overline{\phi}B\widehat{\phi} = (B\cdot\mathrm{D}_\mu)\sigma^\mu. \tag{3.79}$$

Adding and subtracting (3.76) and (3.77) we get:

$$0 = \nabla\cdot\mathrm{D}_\mu,\ \mu = 0,1,2,3, \tag{3.80}$$

$$0 = -w_0 + B\cdot\mathrm{D}_0 + m\rho, \tag{3.81}$$

$$0 = -w_j + B\cdot\mathrm{D}_j,\ j = 1,2,3. \tag{3.82}$$

The four equations (3.80) are the laws of conservation of the D_μ currents. Hence the probability current density is conserved. Multiplying by $\overline{\phi}^{-1}$ on the left side (3.76) is equivalent to (3.62). This is equivalent to the system:

$$0 = i\nabla\eta + qB\eta + m\mathrm{v}\eta, \tag{3.83}$$

$$0 = i\nabla\widehat{\xi} + qB\widehat{\xi} + m\mathrm{v}\widehat{\xi}. \tag{3.84}$$

Thus with v the wave equation (3.76) is:

$$0 = (\nabla\widehat{\phi})i + qB\widehat{\phi} + m\mathrm{v}\widehat{\phi}. \tag{3.85}$$

Since i commutes with σ_1, the multiplication by σ_1 on the right side of this equation changes nothing: **the fourth neutrino-monopole is its own antiparticle**.

3.3 Preserving the mass term

Similar to the previous section, and the previous chapter, a generalization of the mass term is possible for the improved equation. To see this we begin with the simplification of the part of the gauge-invariant derivative of chromodynamics: we consider equally the three $SU(2)$ subgroups of $SU(3)$ by using the potentials:

$$\mathrm{b} = \frac{g_1}{2}B;\ \mathrm{w}^j = \frac{g_2}{2}W^j,\ j = 1, 2, 3. \tag{3.86}$$

$$\begin{aligned}
\mathrm{h}_1^1 &= \frac{g_3}{2}G^1;\ \mathrm{h}_1^2 = \frac{g_3}{2}G^2;\ \mathrm{h}_1^3 - \mathrm{h}_3^3 = \frac{g_3}{2}\Big(-G^3 - \frac{G^8}{\sqrt{3}}\Big),\\
\mathrm{h}_2^1 &= \frac{g_3}{2}G^6;\ \mathrm{h}_2^2 = \frac{g_3}{2}G^7;\ \mathrm{h}_2^3 - \mathrm{h}_1^3 = \frac{g_3}{2}\Big(G^3 - \frac{G^8}{\sqrt{3}}\Big),\\
\mathrm{h}_3^1 &= \frac{g_3}{2}G^4;\ \mathrm{h}_3^2 = -\frac{g_3}{2}G^5;\ \mathrm{h}_3^3 - \mathrm{h}_2^3 = \frac{g_3}{2}\Big(2\frac{G^8}{\sqrt{3}}\Big).
\end{aligned} \tag{3.87}$$

These potentials introduce no supplementary dimension into the gauge group because the sum of $\mathrm{h}_1^3 - \mathrm{h}_3^3$, $\mathrm{h}_2^3 - \mathrm{h}_1^3$ and $\mathrm{h}_3^3 - \mathrm{h}_2^3$ is null. Next we note:

$$\underline{n} = n \mod 3;\ \underline{3} = 3;\ \underline{4} = 1;\ \underline{5} = 2. \tag{3.88}$$

For the gauge-invariant derivative some supplementary terms appear containing the gauge potentials G^k or h_n^p :

$$\frac{g_3}{2}G^k\Lambda_k(\Psi) = \begin{pmatrix} S(\Psi^2) - S(\Psi^3) & S(\Psi^1) - iS(\Psi^4) \\ S(\Psi^1) + iS(\Psi^4) & S(\Psi^2) + S(\Psi_3) \end{pmatrix}, \tag{3.89}$$

$$S(\Psi_1) = 0, \tag{3.90}$$

$$\begin{aligned}
S(\Psi^2) &= \frac{g_3}{2}\Big[(\mathbf{G}^1 - \mathbf{G}^2\mathbf{i})\mathbf{i}\Psi^3 - \mathbf{G}^3\mathbf{i}\Psi^2 + (\mathbf{G}^4 - \mathbf{G}^5\mathbf{i})\mathbf{i}\Psi^4 - \frac{1}{\sqrt{3}}\mathbf{i}\mathbf{G}^8\Psi^2\Big]\\
&= (\mathbf{h}_1^1 - \mathbf{h}_1^2\mathbf{i})\mathbf{i}\Psi^3 - \mathbf{h}_1^3\mathbf{i}\Psi^2 + (\mathbf{h}_3^1 + \mathbf{h}_3^2\mathbf{i})\mathbf{i}\Psi^4 + \mathbf{h}_3^3\mathbf{i}\Psi^2,
\end{aligned} \tag{3.91}$$

$$\begin{aligned}
S(\Psi^3) &= \frac{g_3}{2}\Big[(\mathbf{G}^1+\mathbf{G}^2\mathbf{i})\mathbf{i}\Psi^2+\mathbf{G}^3\mathbf{i}\Psi^3+(\mathbf{G}^6-\mathbf{G}^7\mathbf{i})\mathbf{i}\Psi^4-\frac{1}{\sqrt{3}}\mathbf{G}^8\mathbf{i}\Psi^3\Big]\\
&= (\mathbf{h}_2^1-\mathbf{h}_2^2\mathbf{i})\mathbf{i}\Psi^4-\mathbf{h}_2^3\mathbf{i}\Psi^3+(\mathbf{h}_1^1+\mathbf{h}_1^2\mathbf{i})\mathbf{i}\Psi^2+\mathbf{h}_1^3\mathbf{i}\Psi^3, \qquad (3.92)\\
S(\Psi^4) &= \frac{g_3}{2}\Big[(\mathbf{G}^4+\mathbf{G}^5\mathbf{i})\mathbf{i}\Psi^2+(\mathbf{G}^6+\mathbf{G}^7\mathbf{i})\mathbf{i}\Psi^3+\frac{2}{\sqrt{3}}\mathbf{G}^8\mathbf{i}\Psi^4\Big]\\
&= (\mathbf{h}_3^1-\mathbf{h}_3^2\mathbf{i})\mathbf{i}\Psi^2+\mathbf{h}_3^3\mathbf{i}\Psi^4+(\mathbf{h}_2^1+\mathbf{h}_2^2\mathbf{i})\mathbf{i}\Psi^3+\mathbf{h}_2^3\mathbf{i}\Psi^4, \qquad (3.93)
\end{aligned}$$

The cancellation in (3.90) means that leptons do not experience strong interactions. Next the use of the modulo 3 indices allows us to express the general formula as:

$$S(\Psi^n)=(\mathbf{h}_{n-1}^1-\mathbf{h}_{n-1}^2\mathbf{i})\mathbf{i}\Psi^{\underline{n+1}}-\mathbf{h}_{n-1}^3\mathbf{i}\Psi^n+(\mathbf{h}_{\underline{n+1}}^1+\mathbf{h}_{\underline{n+1}}^2\mathbf{i})\mathbf{i}\Psi^{\underline{n+2}}+\mathbf{h}_{\underline{n+1}}^3\mathbf{i}\Psi^n. \qquad (3.94)$$

We have with (3.50):

$$\begin{aligned}
\underline{D}(\Psi) &= \begin{pmatrix}\mathbf{D}\Psi^2-\mathbf{D}\Psi^3 & \mathbf{D}\Psi^1-i\mathbf{D}\Psi^4\\ \mathbf{D}\Psi_l+i\mathbf{D}\Psi^4 & \mathbf{D}\Psi^2+\mathbf{D}\Psi^3\end{pmatrix},\\
\mathbf{D}\Psi_l &= \boldsymbol{\partial}\Psi_l+\frac{g_1}{2}\mathbf{B}P_0(\Psi_l)+\frac{g_2}{2}\mathbf{W}^jP_j(\Psi_l)=\boldsymbol{\partial}\Psi_l+\mathbf{b}P_0(\Psi_l)+\mathbf{w}^jP_j(\Psi_l),\\
\mathbf{D}\Psi^n &= \boldsymbol{\partial}\Psi^n+\frac{g_1}{2}\mathbf{B}P_0(\Psi^n)+\frac{g_2}{2}\mathbf{W}^jP_j(\Psi^n)+S(\Psi^n) \qquad (3.95)\\
&= \boldsymbol{\partial}\Psi^n+\mathbf{b}P_0(\Psi^n)+\mathbf{w}^jP_j(\Psi^n)+S(\Psi^n).
\end{aligned}$$

We also have with the notations in B.1.2:

$$\begin{aligned}
(\mathbf{h}_{n-1}^1-\mathbf{h}_{n-1}^2\mathbf{i})\mathbf{i}\Psi^{\underline{n+1}} &= i\left(-(\mathrm{h}_{n-1}^1+i\mathrm{h}_{n-1}^2)\overline{\phi}^{3+\underline{n+1}} \quad -(\mathrm{h}_{n-1}^1+i\mathrm{h}_{n-1}^2)\widehat{\phi}^{\underline{n+1}}\right),\\
-\mathbf{h}_{n-1}^3\mathbf{i}\Psi^n &= i\left(\mathrm{h}_{n-1}^3\overline{\phi}^{3+n} \quad \mathrm{h}_{n-1}^3\widehat{\phi}^n\right). \qquad (3.96)
\end{aligned}$$

We next have:

$$\begin{aligned}
(\mathbf{h}_{\underline{n+1}}^1+\mathbf{h}_{\underline{n+1}}^2\mathbf{i})\mathbf{i}\Psi^{\underline{n+2}} &= i\left(-(\mathrm{h}_{\underline{n+1}}^1-i\mathrm{h}_{\underline{n+1}}^2)\overline{\phi}^{3+\underline{n+2}} \quad -(\mathrm{h}_{\underline{n+1}}^1-i\mathrm{h}_{\underline{n+1}}^2)\widehat{\phi}^{\underline{n+2}}\right),\\
\mathbf{h}_{\underline{n+1}}^3\mathbf{i}\Psi^n &= i\left(-\mathrm{h}_{\underline{n+1}}^3\overline{\phi}^{3+n} \quad -\mathrm{h}_{\underline{n+1}}^3\widehat{\phi}^n\right). \qquad (3.97)
\end{aligned}$$

Then if we let

$$S(\Psi^n)=\left(S(\overline{\phi}^{3+n}) \quad S(\widehat{\phi}^n)\right), \qquad (3.98)$$

we get:

$$\begin{aligned}
S(\overline{\phi}^{3+n}) &= -i(\mathrm{h}_{n-1}^1+i\mathrm{h}_{n-1}^2)\overline{\phi}^{3+\underline{n+1}}+i\mathrm{h}_{n-1}^3\overline{\phi}^{3+n}\\
&\quad -i(\mathrm{h}_{\underline{n+1}}^1-i\mathrm{h}_{\underline{n+1}}^2)\overline{\phi}^{3+\underline{n+2}}-i\mathrm{h}_{\underline{n+1}}^3\overline{\phi}^{3+n}, \qquad (3.99)\\
S(\widehat{\phi}^n) &= -i(\mathrm{h}_{n-1}^1+i\mathrm{h}_{n-1}^2)\overline{\phi}^{3+\underline{n+1}}+i\mathrm{h}_{n-1}^3\widehat{\phi}^n\\
&\quad -i(\mathrm{h}_{\underline{n+1}}^1-i\mathrm{h}_{\underline{n+1}}^2)\widehat{\phi}^{\underline{n+2}}-i\mathrm{h}_{\underline{n+1}}^3\widehat{\phi}^n. \qquad (3.100)
\end{aligned}$$

For the part containing the derivative and the electroweak interactions we use equations (3.30) to (3.32). And we use the conjugation $M \mapsto \widehat{M}$ on the right waves, which allows us to get for the gauge-invariant derivative:

$$-i\widehat{D}R^n = -i\widehat{\nabla}R^n + \frac{2}{3}\widehat{\mathrm{b}}R^n + (\widehat{\mathrm{h}}^1_{n-1} - i\widehat{\mathrm{h}}^2_{n-1})R^{\underline{n+1}} + (\widehat{\mathrm{h}}^1_{\underline{n+1}} + i\widehat{\mathrm{h}}^2_{\underline{n+1}})R^{\underline{n+2}}$$
$$- (\widehat{\mathrm{h}}^3_{n-1} - \widehat{\mathrm{h}}^3_{\underline{n+1}})R^n, \quad (3.101)$$
$$-i\overline{D}\widetilde{R}^{3+n} = -i\widehat{\nabla}\widetilde{R}^{3+n} - \frac{4}{3}\widehat{\mathrm{b}}\widetilde{R}^{3+n} + (\widehat{\mathrm{h}}^1_{n-1} - i\widehat{\mathrm{h}}^2_{n-1})\widetilde{R}^{3+\underline{n+1}} \quad (3.102)$$
$$+ (\widehat{\mathrm{h}}^1_{\underline{n+1}} + i\widehat{\mathrm{h}}^2_{\underline{n+1}})\widetilde{R}^{3+\underline{n+2}} - (\widehat{\mathrm{h}}^3_{n-1} - \widehat{\mathrm{h}}^3_{\underline{n+1}})\widetilde{R}^{3+n},$$

$$iD\widehat{L}^n = i\nabla\widehat{L}^n + \frac{\mathrm{b}}{3}\widehat{L}^n + [(\mathrm{w}^1_n + i\mathrm{w}^2_n)\overline{L}^{3+n} - \mathrm{w}^3_n\widehat{L}^n] \quad (3.103)$$
$$+(\mathrm{h}^1_{n-1} + i\mathrm{h}^2_{n-1})\widehat{L}^{\underline{n+1}} + (\mathrm{h}^1_{\underline{n+1}} - i\mathrm{h}^2_{\underline{n+1}})\widehat{L}^{\underline{n+2}} - (\mathrm{h}^3_{n-1} - \mathrm{h}^3_{\underline{n+1}})\widehat{L}^n,$$
$$i\widetilde{D}\overline{L}^{3+n} = i\widetilde{\nabla}\overline{L}^{3+n} + \frac{\mathrm{b}}{3}\overline{L}^{3+n} + [(\mathrm{w}^1_n - i\mathrm{w}^2_n)\widehat{L}^n + \mathrm{w}^3_n\overline{L}^{3+n}] \quad (3.104)$$
$$+(\mathrm{h}^1_{n-1} + i\mathrm{h}^2_{n-1})\overline{L}^{3+\underline{n+1}} + (\mathrm{h}^1_{\underline{n+1}} - i\mathrm{h}^2_{\underline{n+1}})\overline{L}^{3+\underline{n+2}} - (\mathrm{h}^3_{n-1} - \mathrm{h}^3_{\underline{n+1}})\overline{L}^{3+n}.$$

For making this gauge-invariant derivative compatible with the mass term (and we recall that the mass term allows us a direct link with inertia and gravitation), we derive this in a manner completely analogous to that used in 2.2 for the lepton part of the wave. The $m\mathrm{v}\phi\sigma_{12}$ form of the mass term of the improved equation of the electron is conserved. The only thing that changes is the definition of the unitary vector v. We now have indeed twelve chiral currents:

$$\mathrm{D}^n_R = R^n\widetilde{R}^n;\ \mathrm{D}^n_L = L^n\widetilde{L}^n;\ \mathrm{D}^{3+n}_R = \widetilde{R}^{3+n}R^{3+n};\ \mathrm{D}^{3+n}_L = \widetilde{L}^{3+n}L^{3+n}, \quad (3.105)$$

for $n = 2, 3, 4$. The J_q current that replaces J_l is the sum of these twelve chiral currents:

$$\mathrm{J}_q = \sum_{n=2}^{n=4}[\mathrm{D}^n_R + \mathrm{D}^{3+n}_R + \mathrm{D}^n_L + \mathrm{D}^{3+n}_L];\ \rho_q^2 = (\mathrm{J}_q)^2 = \mathrm{J}_q\widehat{\mathrm{J}}_q;\ \mathrm{v}_q = \frac{\mathrm{J}_q}{\rho_q}. \quad (3.106)$$

Since the J_q current is the sum of twelve currents the calculation of the squared scalar product of this vector has twelve squares, all null since each chiral current is on the light cone. And there are $66 = 12 \times 11/2$ scalar products of two distinct currents. Hence the ρ_q^2 term is the sum of 66 relativistically invariant terms:

$$\rho_q^2 = \sum_{n=2}^{n=7} d_n d_n^* + \sum_{n,p,q} s_n^{pq}(s_n^{pq})^*$$
$$d_n = R^n\overline{L}^n + L^n\overline{R}^n = 2\eta^{n\dagger}\xi^n = \mathrm{D}^n_R \cdot \mathrm{D}^n_L, \quad (3.107)$$

where in the s_n^{pq}, $n = 2,3,4,5$, and pq is one of the 15 possible pairs that may be formed with two different numbers taken among $2,3,4,5,6,7$:

$$\begin{aligned}
s_2^{pq} &= 2\eta^{p\dagger}\widehat{\eta}^q = -2\eta^{q\dagger}\widehat{\eta}^p = \mathrm{D}_L^p \cdot \mathrm{D}_L^q, \\
s_3^{pq} &= 2\eta^{q\dagger}\xi^p = \mathrm{D}_R^p \cdot \mathrm{D}_L^q;\; s_4^{pq} = 2\eta^{p\dagger}\xi^q = \mathrm{D}_L^p \cdot \mathrm{D}_R^q, \\
s_5^{pq} &= 2\widehat{\xi}^{p\dagger}\xi^q = -2\widehat{\xi}^{q\dagger}\xi^p = \mathrm{D}_R^p \cdot \mathrm{D}_R^q.
\end{aligned} \tag{3.108}$$

The equations with mass term for the quarks are obtained exactly like the equations of the lepton part (2.76) [45] [50]:

$$\begin{aligned}
0 &= D\widehat{L}^n + q_1 \mathrm{v}_q \widehat{L}^n \sigma_{12};\; 0 = \widehat{D}R^n + q_2 \widehat{\mathrm{v}}_q R^n \sigma_{12}, \\
0 &= D\overline{L}^{3+n} + q_3 \mathrm{v}_q \overline{L}^{3+n} \sigma_{12};\; 0 = \widehat{D}\widetilde{R}^{3+n} + q_4 \widehat{\mathrm{v}}_q \widetilde{R}^{3+n} \sigma_{12},
\end{aligned} \tag{3.109}$$

and always for $n = 2,3,4$. We have four equations in triplicates; thus we consider four proper masses $\hbar c q_j$, $j = 1,2,3,4$. As in 2.2 the mass term $\mathrm{m}\Psi^n$ accounts for the separation of the Ψ^n wave into four parts:

$$\begin{aligned}
\Psi_{dL}^n &= \begin{pmatrix} L^n & 0 \end{pmatrix};\; \Psi_{uL}^n = \begin{pmatrix} 0 & \widetilde{L}^{3+n} \end{pmatrix}, \\
\Psi_{dR}^n &= \begin{pmatrix} R^n & 0 \end{pmatrix};\; \Psi_{uR}^n = \begin{pmatrix} 0 & \widetilde{R}^{3+n} \end{pmatrix}
\end{aligned} \tag{3.110}$$

$$\begin{aligned}
\mathbf{m}(\Psi^n) &= q_1 \Psi_{dL}^n + q_2 \Psi_{dR}^n + q_3 \Psi_{uL}^n + q_4 \Psi_{uR}^n \\
&= \begin{pmatrix} q_1 L^n + q_2 R^n & q_3 \widetilde{L}^{3+n} + q_4 \widetilde{R}^{3+n} \end{pmatrix}.
\end{aligned} \tag{3.111}$$

And we gather these four equations (3.109) together into:

$$0 = \mathbf{D}\Psi^n + \mathbf{v}_q \mathbf{m}(\Psi^n)\gamma_{21};\; \mathbf{v}_q = \begin{pmatrix} 0 & \mathrm{v}_q \end{pmatrix};\; \mathbf{v}_q^2 = 1. \tag{3.112}$$

Next letting:

$$\underline{M}(\Psi) = \begin{pmatrix} \mathbf{v}_q[\mathbf{m}(\Psi^2) - \mathbf{m}(\Psi^3)] & \mathbf{v}\mathbf{m}(\Psi^1) - i\mathbf{v}_q\mathbf{m}(\Psi^4) \\ \mathbf{v}\mathbf{m}(\Psi^1) + i\mathbf{v}_q\mathbf{m}(\Psi^4) & \mathbf{v}_q[\mathbf{m}(\Psi^2) + \mathbf{m}(\Psi_3)] \end{pmatrix}, \tag{3.113}$$

The wave equation that generalizes the improved equation of the electron is then expressed as

$$0 = \underline{D}\Psi\Gamma_{012} + \underline{M}(\Psi)\Gamma_0, \tag{3.114}$$

while the equation invariant under Cl_3^* (we recall that this invariance automatically implies relativistic invariance) is obtained simply by multiplying on the left side by the reverse:

$$0 = \widetilde{\Psi}\underline{D}\Psi\Gamma_{012} + \widetilde{\Psi}\underline{M}(\Psi)\Gamma_0. \tag{3.115}$$

It is the strict link between the reversion in $Cl_{3,3}$ and the reversion in $Cl_{1,3}$ which enables the complete separation of the mass term of the lepton part from the quark part. This strict link is nontrivial and is established in B.2 .

Moreover except in a very particular case, $\Psi(\mathrm{x})$ is invertible. We may thus derive the invariant form (3.115) from (3.114) by multiplying on the left side by $\widetilde{\Psi}$. And multiplying the left side of (3.115) by $\widetilde{\Psi}^{-1}$, we can derive the usual (3.114) form. We recall that this justifies, for all lepton waves, that we are able to derive the wave equations from the Lagrangian density. Then the same behavior is observed for the quark waves.

3.4 Invariance

The invariance of the (3.109) equations is similar to that of the leptonic wave studied in 2.3. For the form invariance that includes relativistic invariance, it is enough to add to (2.81) and (2.82), and to the covariance of the b and w^j potentials that of the g_3G^k or h^m that are derived from (3.87):

$$\mathrm{h}^m_n = \overline{M}\mathrm{h}'^m_n\widehat{M};\ q_n = rq'_n;\ r = |\det(M)|. \tag{3.116}$$

We derive:

$$\begin{aligned}
&0 = D'\widehat{L'}^n + q'_1\mathrm{v}'_q\widehat{L'}^n\sigma_{12};\ 0 = \widehat{D}'R'^n + q'_2\widehat{\mathrm{v}}'_qR'^n\sigma_{12},\\
&0 = D'\overline{L}'^{3+n} + q'_3\mathrm{v}'_q\overline{L}'^{3+n}\sigma_{12};\ 0 = \widehat{D}'\widetilde{R}'^{3+n} + q'_4\widehat{\mathrm{v}}'_q\widetilde{R}'^{3+n}\sigma_{12},
\end{aligned} \tag{3.117}$$

which implies the form invariance of the wave equations.

The gauge invariance under the $U(1)$ group generated by $\underline{P}_0$ results from the equalities (2.93)–(2.94) in which it is enough to replace P_0 by $\underline{P}_0$ with the Ψ in (3.1). What changes from the lepton case comes only with $\underline{P}_0$ which gives:

$$\begin{aligned}
\underline{P}_0(\Psi) &= \begin{pmatrix} P_0(\Psi^1) & P_0(\Psi^2)\\ P_0(\Psi^3) & P_0(\Psi^4)\end{pmatrix};\ P_0(\Psi^n) = -\frac{1}{3}\Psi^n\gamma_{21} + \frac{1}{2}(\Psi^n\mathbf{i} + \mathbf{i}\Psi^n\gamma_{30}),\\
P_0(\Psi^n) &= -\frac{i}{3}\begin{pmatrix} R^n - L^n & \widetilde{R}^{3+n} - \widetilde{L}^{3+n}\end{pmatrix} + i\begin{pmatrix} R^n & -\widetilde{R}^{3+n}\end{pmatrix}\\
&= \begin{pmatrix} \frac{2i}{3}R^n + \frac{i}{3}L^n & -\frac{4i}{3}\widetilde{R}^{3+n} + \frac{i}{3}\widetilde{L}^{3+n}\end{pmatrix},\ n = 2,3,4.
\end{aligned} \tag{3.118}$$

We then have:

$$\begin{aligned}
\Psi'^n &= [\exp(a^0P_0)](\Psi^n) = \begin{pmatrix} R'^n + L'^n & \widetilde{R}'^{3+n} + \widetilde{L}'^{3+n}\end{pmatrix},\\
R'^n &= e^{2ia^0/3}R^n;\ L'^n = e^{ia^0/3}L^n,\\
\widetilde{R}'^{3+n} &= e^{-4ia^0/3}\widetilde{R}^{3+n};\ \widetilde{L}'^{3+n} = e^{ia^0/3}\widetilde{L}^{3+n}.
\end{aligned} \tag{3.119}$$

All left waves turn with the same angle $a^0/3$, and only the left waves have this property. This is how they come to be invariant under the $SU(2)$ gauge group mixing the different left waves. We get:

$$\mathrm{D}'^n_R = R'^n\widetilde{R}'^n = e^{2ia^0/3}R^ne^{-2ia^0/3}\widetilde{R}^n = R^n\widetilde{R}^n = \mathrm{D}^n_R. \tag{3.120}$$

And similarly we have:

$$\mathrm{D}'^{n}_{L} = \mathrm{D}^{n}_{L};\ \mathrm{D}'^{3+n}_{L} = \mathrm{D}^{3+n}_{L};\ \mathrm{D}'^{3+n}_{R} = \mathrm{D}^{3+n}_{R};\ \mathrm{J}'_q = \mathrm{J}_q;\ \mathrm{v}'_q = \mathrm{v}_q, \qquad (3.121)$$

and so the mass terms of the wave equations are invariant under the $U(1)$ gauge group. As in the case of the leptonic wave all left waves transform in the same manner: this is what is responsible for the commutation between the $\underline{P}_0$ operator and the three $\underline{P}_j$, $j = 1,2,3$. To study the other parts of the gauge group we start from (3.95), so that for $n = 2$ and with (3.91) we have

$$\begin{aligned}
\mathbf{D}\Psi^2 &= \boldsymbol{\partial}\Psi^2 + \mathbf{b}P_0(\Psi^2) + \mathbf{w}^j P_j(\Psi^2) + S(\Psi^2)\\
&= \boldsymbol{\partial}\Psi^2 + \mathbf{b}P_0(\Psi^2) + \mathbf{w}^j P_j(\Psi^2) \qquad (3.122)\\
&+ (\mathbf{h}^1_2 - \mathbf{h}^2_2\mathbf{i})\mathbf{i}\Psi^4 - \mathbf{h}^3_2\mathbf{i}\Psi^3 + (\mathbf{h}^1_1 + \mathbf{h}^2_1\mathbf{i})\mathbf{i}\Psi^2 + \mathbf{h}^3_1\mathbf{i}\Psi^3.
\end{aligned}$$

With (3.1) to (3.18) the previous equation is equivalent to the system:

$$\begin{aligned}
iD\overline{\phi}^5 &= i\nabla\overline{\phi}^5 - \mathrm{b}(\frac{4}{3}\overline{R}^5 + \frac{1}{3}\overline{L}^5) + [(\mathrm{w}^1_2 - i\mathrm{w}^2_2)\widehat{L}^2 + \mathrm{w}^3_2\overline{L}^5]\\
&+ [(\mathrm{h}^1_1 + i\mathrm{h}^2_1)\overline{\phi}^6 - \mathrm{h}^3_1\overline{\phi}^5] + [(\mathrm{h}^1_3 - i\mathrm{h}^2_3)\overline{\phi}^7 + \mathrm{h}^3_3\overline{\phi}^5], \qquad (3.123)\\
iD\widehat{\phi}^2 &= i\nabla\widehat{\phi}^2 + \mathrm{b}(\frac{2}{3}\widehat{R}^2 + \frac{1}{3}\widehat{L}^5) + [(\mathrm{w}^1_2 + i\mathrm{w}^2_2)\overline{L}^5 - \mathrm{w}^3_2\widehat{L}^2]\\
&+ [(\mathrm{h}^1_1 + i\mathrm{h}^2_1)\widehat{\phi}^3 - \mathrm{h}^3_1\widehat{\phi}^2] + [(\mathrm{h}^1_3 - i\mathrm{h}^2_3)\widehat{\phi}^4 + \mathrm{h}^3_3\widehat{\phi}^2],
\end{aligned}$$

There are two other equations when using the matrix representation of $Cl_{1,3}$, which are equivalent to the two previous ones and result from the main automorphism $P : M \mapsto \widehat{M}$. Next, by using this automorphism for the right waves we get the equivalent system:

$$\begin{aligned}
iD\eta^5 &= i\nabla\eta^5 + \frac{\mathrm{b}}{3}\eta^5 + [(\mathrm{w}^1_2 - i\mathrm{w}^2_2)\eta^2 + \mathrm{w}^3_2\eta^5]\\
&+ [(\mathrm{h}^1_1 + i\mathrm{h}^2_1)\eta^6 - \mathrm{h}^3_1\eta^5] + [(\mathrm{h}^1_3 - i\mathrm{h}^2_3)\eta^7 + \mathrm{h}^3_3\eta^5],\\
iD\eta^2 &= i\nabla\eta^2 + \frac{\mathrm{b}}{3}\eta^2 + [(\mathrm{w}^1_2 + i\mathrm{w}^2_2)\eta^5 - \mathrm{w}^3_2\eta^2]\\
&+ [(\mathrm{h}^1_1 + i\mathrm{h}^2_1)\eta^3 - \mathrm{h}^3_1\eta^2] + [(\mathrm{h}^1_3 - i\mathrm{h}^2_3)\eta^4 + \mathrm{h}^3_3\eta^2], \qquad (3.124)\\
-i\widehat{D}\xi^2 &= -i\widehat{\nabla}\xi^2 + \frac{2}{3}\widehat{\mathrm{b}}\xi^2 + [(\widehat{\mathrm{h}}^1_1 - i\widehat{\mathrm{h}}^2_1)\xi^3 - \widehat{\mathrm{h}}^3_1\xi^2] + [(\widehat{\mathrm{h}}^1_3 + i\widehat{\mathrm{h}}^2_3)\xi^4 + \widehat{\mathrm{h}}^3_3\xi^2],\\
-i\widehat{D}\xi^5 &= -i\widehat{\nabla}\xi^5 - \frac{4}{3}\widehat{\mathrm{b}}\xi^5 + [(\widehat{\mathrm{h}}^1_1 - i\widehat{\mathrm{h}}^2_1)\xi^6 - \widehat{\mathrm{h}}^3_1\xi^5] + [(\widehat{\mathrm{h}}^1_3 + i\widehat{\mathrm{h}}^2_3)\xi^7 + \widehat{\mathrm{h}}^3_3\xi^5].
\end{aligned}$$

Next we have two other systems with the same structure which are obtained by circularly permuting the indices $2,3,4$ and $5,6,7$ (corresponding to the

r, g, b colors of the quarks) everywhere these indices are present:

$$
\begin{aligned}
iD\eta^6 &= i\nabla\eta^6 + \frac{\mathrm{b}}{3}\eta^6 + [(\mathrm{w}_3^1 - i\mathrm{w}_3^2)\eta^3 + \mathrm{w}_3^3\eta^6] \\
&\quad + [(\mathrm{h}_2^1 + i\mathrm{h}_2^2)\eta^7 - \mathrm{h}_2^3\eta^6] + [(\mathrm{h}_1^1 - i\mathrm{h}_1^2)\eta^5 + \mathrm{h}_1^3\eta^6], \\
iD\eta^3 &= i\nabla\eta^3 + \frac{\mathrm{b}}{3}\eta^3 + [(\mathrm{w}_3^1 + i\mathrm{w}_3^2)\eta^6 - \mathrm{w}_3^3\eta^3] \\
&\quad + [(\mathrm{h}_2^1 + i\mathrm{h}_2^2)\eta^4 - \mathrm{h}_2^3\eta^3] + [(\mathrm{h}_1^1 - i\mathrm{h}_1^2)\eta^2 + \mathrm{h}_1^3\eta^3], \qquad (3.125) \\
-i\widehat{D}\xi^3 &= -i\widehat{\nabla}\xi^3 + \frac{2}{3}\widehat{\mathrm{b}}\xi^3 + [(\widehat{\mathrm{h}}_2^1 - i\widehat{\mathrm{h}}_2^2)\xi^4 - \widehat{\mathrm{h}}_2^3\xi^3] + [(\widehat{\mathrm{h}}_1^1 + i\widehat{\mathrm{h}}_1^2)\xi^2 + \widehat{\mathrm{h}}_1^3\xi^3], \\
-i\widehat{D}\xi^6 &= -i\widehat{\nabla}\xi^6 - \frac{4}{3}\widehat{\mathrm{b}}\xi^6 + [(\widehat{\mathrm{h}}_2^1 - i\widehat{\mathrm{h}}_2^2)\xi^7 - \widehat{\mathrm{h}}_2^3\xi^6] + [(\widehat{\mathrm{h}}_1^1 + i\widehat{\mathrm{h}}_1^2)\xi^5 + \widehat{\mathrm{h}}_1^3\xi^6]. \\
iD\eta^7 &= i\nabla\eta^7 + \frac{\mathrm{b}}{3}\eta^7 + [(\mathrm{w}_1^1 - i\mathrm{w}_1^2)\eta^4 + \mathrm{w}_1^3\eta^7] \\
&\quad + [(\mathrm{h}_3^1 + i\mathrm{h}_3^2)\eta^5 - \mathrm{h}_3^3\eta^7] + [(\mathrm{h}_2^1 - i\mathrm{h}_2^2)\eta^6 + \mathrm{h}_2^3\eta^7], \\
iD\eta^4 &= i\nabla\eta^4 + \frac{\mathrm{b}}{3}\eta^4 + [(\mathrm{w}_1^1 + i\mathrm{w}_1^2)\eta^7 - \mathrm{w}_1^3\eta^4] \\
&\quad + [(\mathrm{h}_3^1 + i\mathrm{h}_3^2)\eta^2 - \mathrm{h}_3^3\eta^4] + [(\mathrm{h}_2^1 - i\mathrm{h}_2^2)\eta^3 + \mathrm{h}_2^3\eta^4], \\
-i\widehat{D}\xi^4 &= -i\widehat{\nabla}\xi^4 + \frac{2}{3}\widehat{\mathrm{b}}\xi^4 + [(\widehat{\mathrm{h}}_3^1 - i\widehat{\mathrm{h}}_3^2)\xi^2 - \widehat{\mathrm{h}}_3^3\xi^4] + [(\widehat{\mathrm{h}}_2^1 + i\widehat{\mathrm{h}}_2^2)\xi^3 + \widehat{\mathrm{h}}_2^3\xi^4], \\
-i\widehat{D}\xi^7 &= -i\widehat{\nabla}\xi^7 - \frac{4}{3}\widehat{\mathrm{b}}\xi^7 + [(\widehat{\mathrm{h}}_3^1 - i\widehat{\mathrm{h}}_3^2)\xi^5 - \widehat{\mathrm{h}}_3^3\xi^7] + [(\widehat{\mathrm{h}}_2^1 + i\widehat{\mathrm{h}}_2^2)\xi^6 + \widehat{\mathrm{h}}_2^3\xi^7].
\end{aligned}
$$

The invariance under the $SU(2)$ group is the same as what we saw for the leptonic wave. This invariance actually results from:

$$
\begin{aligned}
&\mathrm{D}_L^{n,\,3+n} - i d_L^{n,\,3+n} = 2L^n L^{3+n};\ \mathrm{D}_L^n = L^n \widetilde{L}^n;\ \mathrm{D}_L^{3+n} = \widetilde{L}^{3+n} L^{3+n}, \\
&\qquad \mathrm{w}_n^1 = \frac{g_2}{2}\mathrm{D}_L^{n,\,3+n};\ \mathrm{w}_n^2 = \frac{g_2}{2} d_L^{n,\,3+n};\ \mathrm{w}_n^3 = \frac{g_2}{2}(\mathrm{D}_L^{3+n} - \mathrm{D}_L^n), \qquad (3.126)
\end{aligned}
$$

which are enough to obtain the gauge invariance, as we saw in 2.3.2. And this gives the $U(1) \times SU(2)$ structure of the electroweak gauge group. The only difference from the Standard Model is that we do not need to postulate this result: **we derive the structure from the operators themselves**.

For the $SU(2)$ part of the electroweak gauge group, and since the invariance has exactly the same form as in 2.3.2 , we obtain the following, using the same identities as (2.140) and (2.141) (a detailed calculation is in D.3):

$$
\begin{aligned}
(W_n^1 + iW_n^2)\overline{L}^{3+n} - W_n^3\widehat{L}^n &= -3\widetilde{L}^{3+n} L^{3+n}\widehat{L}^n = -3\mathrm{D}_L^{3+n}\widehat{L}^n = -3W_n^3\widehat{L}^n, \\
(W_n^1 - iW_n^2)\widehat{L}^n + W_n^3\overline{L}^{3+n} &= -3L^n\widetilde{L}^n\overline{L}^{3+n} = -3\mathrm{D}_L^n\overline{L}^{3+n} = 3W_n^3\overline{L}^{3+n}. \qquad (3.127)
\end{aligned}
$$

In the case of the quarks we have moreover the same formula of transformation for $n = 2, 3, 4$; this gives the commutation between the $\underline{P}_n$ and the Λ^k operators of the group of chromodynamics, which act only on the n index, thus giving rise to the $U(1) \times SU(2) \times SU(3)$ structure of the gauge group of the Standard Model. The gauge invariance under the $SU(3)$ group gives a simplification of the wave equations with proper mass (see D.3). Our improved equations have the form:

$$
\begin{aligned}
0 &= D\widehat{L}^n + iq_1\mathrm{v}_q\widehat{L}^n;\ 0 = \widehat{D}R^n + iq_2\widehat{\mathrm{v}}_q R^n,\\
0 &= \widetilde{D}\overline{L}^{3+n} + iq_3\mathrm{v}_q\overline{L}^{3+n};\ 0 = \overline{D}\widetilde{R}^{3+n} + iq_4\widehat{\mathrm{v}}_q\widetilde{R}^{3+n},
\end{aligned} \tag{3.128}
$$

$$
D\widehat{L}^n = \sigma^\mu\Big[\partial_\mu + i\Big(-\frac{\mathrm{b}_\mu}{3} + 3\mathrm{w}^3_{n\mu} - 3\mathrm{h}^{d3}_{L\underline{n+1}\mu} + 3\mathrm{h}^{d3}_{L\,n-1\,\mu}\Big)\Big]\widehat{L}^n, \tag{3.129}
$$

$$
\widehat{D}R^n = \widehat{\sigma}^\mu\Big[\partial_\mu + i\Big(\frac{2\mathrm{b}_\mu}{3} + 3\mathrm{h}^{d3}_{R\underline{n+1}\mu} - 3\mathrm{h}^{d3}_{R\,n-1\,\mu}\Big)\Big]R^n, \tag{3.130}
$$

$$
\widetilde{D}\overline{L}^{3+n} = \sigma^\mu\Big[\partial_\mu + i\Big(-\frac{\mathrm{b}_\mu}{3} - 3\mathrm{w}^3_{n\mu} - 3\mathrm{h}^{u3}_{L\underline{n+1}\mu} + 3\mathrm{h}^{u3}_{L\,n-1\,\mu}\Big)\Big]\overline{L}^{3+n}, \tag{3.131}
$$

$$
\overline{D}\widetilde{R}^{3+n} = \widehat{\sigma}^\mu\Big[\partial_\mu + i\Big(-\frac{4\mathrm{b}_\mu}{3} + 3\mathrm{h}^{u3}_{R\underline{n+1}\mu} - 3\mathrm{h}^{u3}_{R\,n-1\,\mu}\Big)\Big]\widetilde{R}^{3+n}. \tag{3.132}
$$

Here the w potentials depend on color and moreover the h potentials have a double dependence: their two indices with value $2, 3, 4$ come from the generators of the $SU(3)$ group, while their indices L, R and d, u are linked with the spinors on which they act. Then the wave equations that are used to obtain the Lagrangian density are the equations governing right and left waves.

3.5 Wave equation – Lagrangian density

We multiply the wave equations (3.128) of η^n by $-i\mathbf{m}\eta^{n\dagger}/q_1$, the wave equations of ξ^n by $-i\mathbf{m}\xi^{n\dagger}/q_2$, the wave equations of η^{3+n} by $-i\mathbf{m}\eta^{3+n\dagger}/q_3$ and finally the wave equations of ξ^{3+n} by $-i\mathbf{m}\xi^{3+n\dagger}/q_4$, always by the left side. For the lepton part we saw in Chapter 2 how the Lagrangian density of the electron wave is generalized for several Lagrangian densities coming from the different wave equations. Among these densities, $\mathcal{L}_q^+$ is obtained as the sum of the different real parts, and $\mathcal{L}_q^-$ is obtained as the difference between the real parts coming from the left waves and the right waves. Since the ρ_q density is calculated like ρ_l we will get similar results. We let:

$$
\mathcal{L}_q^+ = \sum_{n=2}^{4}\left[\begin{array}{l}\frac{m}{km_1}\eta^{n\dagger}\sigma^\mu(-i\partial_\mu + d^1_{n\mu}) + \frac{m}{km_2}\xi^{n\dagger}\widehat{\sigma}^\mu(-i\partial_\mu + d^2_{n\mu})\\ +\frac{m}{km_3}\eta^{3+n\dagger}\sigma^\mu(-i\partial_\mu + d^3_{n\mu}) + \frac{m}{km_4}\xi^{3+n\dagger}\widehat{\sigma}^\mu(-i\partial_\mu + d^4_{n\mu})\end{array}\right], \tag{3.133}
$$

where we may set:

$$
\begin{aligned}
d^1_\mu &= -\frac{\mathrm{b}_\mu}{3} + 3\mathrm{w}^3_{n\mu} - 3\mathrm{h}^{d3}_{L\underline{n+1}\mu} + 3\mathrm{h}^{d3}_{L\,n-1\,\mu} + q_1\mathrm{v}_{q\mu},\\
d^2_\mu &= \frac{2\mathrm{b}_\mu}{3} + 3\mathrm{h}^{d3}_{R\underline{n+1}\mu} - 3\mathrm{h}^{d3}_{R\,n-1\,\mu} + q_2\mathrm{v}_{q\mu},
\end{aligned} \tag{3.134}
$$

$$d^3_\mu = -\frac{\mathrm{b}_\mu}{3} - 3\mathrm{w}^3_{n\mu} - 3\mathrm{h}^{u3}_{L\underline{n+1}\mu} + 3\mathrm{h}^{u3}_{L\,n-1\,\mu} + q_3\mathrm{v}_{q\mu},$$
$$d^4_\mu = -\frac{4\mathrm{b}_\mu}{3} + 3\mathrm{h}^{u3}_{R\underline{n+1}\mu} - 3\mathrm{h}^{u3}_{R\,n-1\,\mu} + q_4\mathrm{v}_{q\mu}, \tag{3.135}$$

The Lagrangian densities $\mathcal{L}^+_q$ and $\mathcal{L}^-_q$ satisfy:

$$0 = k\mathcal{L}^+_q = \sum_{n=2}^{4}\begin{bmatrix} \frac{\mathbf{m}}{q_1}(-i\eta^{n\dagger}\sigma^\mu\partial_\mu\eta^n + d^1_\mu\mathrm{D}^{n\mu}_L) + \frac{\mathbf{m}}{q_2}(-i\xi^{n\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^n + d^2_\mu\mathrm{D}^{n\mu}_R) \\ +\frac{\mathbf{m}}{q_3}(-i\eta^{3+n\dagger}\sigma^\mu\partial_\mu\eta^{3+n} + d^3_\mu\mathrm{D}^{3+n\mu}_L) \\ +\frac{\mathbf{m}}{q_4}(-i\xi^{3+n\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^{3+n} + d^4_\mu\mathrm{D}^{3+n\mu}_R)\end{bmatrix}, \tag{3.136}$$

$$0 = k\mathcal{L}^-_q = \sum_{n=2}^{4}\begin{bmatrix} -\frac{\mathbf{m}}{q_1}(-i\eta^{n\dagger}\sigma^\mu\partial_\mu\eta^n + d^1_\mu\mathrm{D}^{n\mu}_L) + \frac{\mathbf{m}}{q_2}(-i\xi^{n\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^n + d^2_\mu\mathrm{D}^{n\mu}_R) \\ -\frac{\mathbf{m}}{q_3}(-i\eta^{3+n\dagger}\sigma^\mu\partial_\mu\eta^{3+n} + d^3_\mu\mathrm{D}^{3+n\mu}_L) \\ +\frac{\mathbf{m}}{q_4}(-i\xi^{3+n\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^{3+n} + d^4_\mu\mathrm{D}^{3+n\mu}_R)\end{bmatrix}. \tag{3.137}$$

The fact that these Lagrangian densities are null at each point of space-time is due to their construction from (3.129) to (3.132). Moreover these tensor densities are real because their imaginary part is null. We see this for instance in (3.129) which for $n = 2$ gives:

$$0 = \eta^{2\dagger}\sigma^\mu[-i\partial_\mu - \mathrm{b}_\mu/3 + 3\mathrm{w}^3_{2\mu} - 3\mathrm{h}^{d3}_{L3\mu} + 3\mathrm{h}^{d3}_{L1\mu} + q_1\mathrm{v}_\mu]\eta^2, \tag{3.138}$$
$$0 = [i\partial_\mu\eta^{2\dagger}\sigma^\mu\eta^2 + \eta^{2\dagger}(-\mathrm{b}_\mu/3 + 3\mathrm{w}^3_{2\mu} - 3\mathrm{h}^{d3}_{L3\mu} + 3\mathrm{h}^{d3}_{L1\mu} + q_1\mathrm{v}_\mu)]\eta^2.$$

Then subtracting we get:

$$0 = -i(\eta^{2\dagger}\sigma^\mu\partial_\mu\eta^2 + \partial_\mu\eta^{2\dagger}\sigma^\mu\eta^2) = -i\partial_\mu(\eta^{2\dagger}\sigma^\mu\eta^2);\ 0 = \partial_\mu\mathrm{D}^{2\mu}_L. \tag{3.139}$$

The left current D^2_L is thus conservative. It is the same for the different currents: all the left or right currents are conservative and on the light cone. We now calculate the mass term of the $\mathcal{L}^+_q$ density. We get:

$$\mathbf{m}\sum_{n=2}^{4}(\mathrm{D}^{n\mu}_L + \mathrm{D}^{n\mu}_R + \mathrm{D}^{3+n\mu}_L + \mathrm{D}^{3+n\mu}_R)\mathrm{v}_{q\mu} = \mathbf{m}\mathrm{J}^\mu_q\mathrm{v}_{q\mu} = \mathbf{m}\rho_q\mathrm{v}^2_q = \mathbf{m}\rho_q. \tag{3.140}$$

As with the lepton case we may consider the d^k vectors in (3.134) as a sum of a part g^k_n coming from the gauge terms and a part $q_k\mathrm{v}_q$ resulting from inertia. We then replace d^k by:

$$d^k = g^k_n + q_k\mathrm{v}_q;\ \mathrm{D}^k_{n\mu} = \partial_\mu + i(g^k_{n\mu} + q_k\mathrm{v}_{q\mu});\ a\delta^k_{n\mu}b = a(\mathrm{D}^k_{n\mu}b) - (\mathrm{D}^{k*}_{n\mu}a)b. \tag{3.141}$$

And we have:

$$g^1_n = -\frac{\mathrm{b}}{3} + 3\mathrm{w}^3_n - 3\mathrm{h}^{d3}_{L\underline{n+1}} + 3\mathrm{h}^{d3}_{L\,n-1};\ g^2_n = \frac{2\mathrm{b}}{3} + 3\mathrm{h}^{d3}_{R\underline{n+1}} - 3\mathrm{h}^{d3}_{R\,n-1},$$

$$g_n^3 = -\frac{\mathrm{b}}{3} - 3\mathrm{w}_n^3 - 3\mathrm{h}^{u3}_{L\underline{n+1}} + 3\mathrm{h}^{u3}_{L\,n-1};\ g_n^4 = -\frac{4\mathrm{b}}{3} + 3\mathrm{h}^{u3}_{R\underline{n+1}} - 3\mathrm{h}^{u3}_{R\,n-1}. \tag{3.142}$$

$$k\mathcal{L}_q^+ = \frac{1}{2}\sum_{n=2}^{4}\begin{bmatrix} \frac{\mathbf{m}}{q_1}\eta^{n\dagger}\sigma^\mu\delta^1_{n\mu}\eta^n + \frac{\mathbf{m}}{q_2}\xi^{n\dagger}\widehat{\sigma}^\mu\delta^2_{n\mu}\xi^n \\ +\frac{\mathbf{m}}{q_3}\eta^{3+n\dagger}\sigma^\mu\delta^3_{n\mu}\eta^{3+n} + \frac{\mathbf{m}}{q_4}\xi^{3+n\dagger}\widehat{\sigma}^\mu\delta^4_{n\mu}\xi^{3+n}\end{bmatrix}, \tag{3.143}$$

$$k\mathcal{L}_q^- = \frac{1}{2}\sum_{n=2}^{4}\begin{bmatrix} -\frac{\mathbf{m}}{q_1}\eta^{n\dagger}\sigma^\mu\delta^1_{n\mu}\eta^n + \frac{\mathbf{m}}{q_2}\xi^{n\dagger}\widehat{\sigma}^\mu\delta^2_{n\mu}\xi^n \\ -\frac{\mathbf{m}}{q_3}\eta^{3+n\dagger}\sigma^\mu\delta^3_{n\mu}\eta^{3+n} + \frac{\mathbf{m}}{q_4}\xi^{3+n\dagger}\widehat{\sigma}^\mu\delta^4_{n\mu}\xi^{3+n}\end{bmatrix}. \tag{3.144}$$

The Lagrangian density $\mathcal{L}_q^+$ is a sum of twelve terms, with the same structure as the four terms that we had in Chapter 2 for the leptonic wave. We may thus replicate what we detailed in 2.3.4. Since there we used only the algebraic properties of multiplication in Cl_3, we can easily redo with η^n what we proved with η^1. Moreover, ξ^n acts like ξ^1, η^{3+n} acts like η^8 and ξ^{3+n} acts like ξ^8. Thus the wave equations allow us to arrive at $0 = \mathcal{L}_q^+$, and moreover the Lagrange equations, without any supplementary condition, allow us to obtain for each spinor the real equations equivalent to the wave equation expressed in Cl_3. When we vary the Lagrangian density in relation to the variables contained in η^2, the gauge potentials introduce no supplementary term. This comes from the mechanism described in 2.3.4 for the b potential, as well as for the other potentials, because w_3^3 acts on η^2 only by the term D_L^5. This is the same for the potentials of chromodynamics. The h^3_{L3} potential acts on η^2 only by the D_L^3 term, and h^3_{L1} acts on the η^2 term only by the D_L^4 term.

About the antiparticles we may also use without any change what we said about the electron in Chapter 1 and about the lepton wave in Chapter 2. The only change in this passing to the "anti-world" is the replacement of the ∂_μ by $-\partial_\mu$ and the exchange of η and ξ. The double link between the wave equation and the Lagrangian density is totally conserved.

3.6 Energy–momentum tensors

Here as well we again obtain what we learned from the Dirac equation and from its extension to the lepton wave: the existence of not only one Lagrangian density and one energy–momentum tensor associated to the invariance under translation, but at least two tensors that we must study. With the g_n^k in (3.142) and with the mass term (3.140) we may read the Lagrangian density $\mathcal{L}_q^+$ in (3.143) as:

$$k\mathcal{L}_q^+ = \mathbf{m}\rho_q + \sum_{n=2}^{4}\begin{bmatrix} \frac{\mathbf{m}}{q_1}\Re(-i\eta^{n\dagger}\sigma^\mu\partial_\mu\eta^n) + \frac{\mathbf{m}}{q_1}g^1_{n\mu}\eta^{n\dagger}\sigma^\mu\eta^n \\ \frac{\mathbf{m}}{q_2}\Re(-i\xi^{n\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^n) + \frac{\mathbf{m}}{q_2}g^2_{n\mu}\xi^{n\dagger}\widehat{\sigma}^\mu\xi^n \\ \frac{\mathbf{m}}{q_3}\Re(-i\eta^{3+n\dagger}\sigma^\mu\partial_\mu\eta^{3+n}) + \frac{\mathbf{m}}{q_3}g^3_{n\mu}\eta^{3+n\dagger}\sigma^\mu\eta^{3+n} \\ \frac{\mathbf{m}}{q_4}\Re(-i\xi^{3+n\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^{3+n}) + \frac{\mathbf{m}}{q_4}g^4_{n\mu}\xi^{3+n\dagger}\widehat{\sigma}^\mu\xi^{3+n}\end{bmatrix}. \tag{3.145}$$

This Lagrangian density is the sum of twelve terms. It is invariant under space-time translations; thus a conservative tensor density of energy–momentum is associated, the sum of twelve densities:

$$
\begin{aligned}
T &= \sum_{n=2}^{4}\Big(\frac{\mathbf{m}}{kq_1}T_L^n + \frac{\mathbf{m}}{kq_2}T_R^n + \frac{\mathbf{m}}{kq_3}T_L^{3+n} + \frac{\mathbf{m}}{kq_4}T_R^{3+n}\Big), && (3.146)\\
T_{L\lambda}^{n\mu} &= \Re(i\eta^{n\dagger}\sigma^\mu\partial_\lambda\eta^n) - g_{n\lambda}^1\eta^{n\dagger}\sigma^\mu\eta^n, && (3.147)\\
T_{R\lambda}^{n\mu} &= \Re(i\xi^{n\dagger}\widehat{\sigma}^\mu\partial_\lambda\xi^n) - g_{n\lambda}^2\xi^{n\dagger}\widehat{\sigma}^\mu\xi^n, && (3.148)\\
T_{L\lambda}^{3+n\mu} &= \Re(i\eta^{3+n\dagger}\sigma^\mu\partial_\lambda\eta^{3+n}) - g_{n\lambda}^3\eta^{3+n\dagger}\sigma^\mu\eta^{3+n}, && (3.149)\\
T_{R\lambda}^{3+n\mu} &= \Re(i\xi^{3+n\dagger}\widehat{\sigma}^\mu\partial_\lambda\xi^{3+n}) - g_{n\lambda}^4\xi^{3+n\dagger}\widehat{\sigma}^\mu\xi^{3+n}. && (3.150)
\end{aligned}
$$

In particular, for the T_0^0 component we have:

$$
\begin{aligned}
T_0^0 &= \sum_{n=2}^{4}\Big(\frac{\mathbf{m}}{kq_1}T_{L0}^{n0} + \frac{\mathbf{m}}{kq_2}T_{R0}^{n0} + \frac{\mathbf{m}}{kq_3}T_{L0}^{3+n0} + \frac{\mathbf{m}}{kq_4}T_{R0}^{3+n0}\Big), && (3.151)\\
T_{L0}^{n0} &= \Re(i\eta^{n\dagger}\partial_0\eta^n) - g_{n0}^1\eta^{n\dagger}\eta^n, && (3.152)\\
T_{R0}^{n0} &= \Re(i\xi^{n\dagger}\partial_0\xi^n) - g_{n0}^2\xi^{n\dagger}\xi^n, && (3.153)\\
T_{L0}^{3+n0} &= \Re(i\eta^{3+n\dagger}\partial_0\eta^{3+n}) - g_{n0}^3\eta^{3+n\dagger}\eta^{3+n}, && (3.154)\\
T_{R0}^{3+n0} &= \Re(i\xi^{3+n\dagger}\partial_\mu\xi^{3+n}) - g_{n0}^4\xi^{3+n\dagger}\xi^{3+n}. && (3.155)
\end{aligned}
$$

For a solution to the wave equation with an energy E of the whole wave, we have:

$$
\begin{aligned}
-id_\mu &= -i\partial_\mu + g_{n\mu}^k,\\
-id_0\xi^n &= \frac{E}{\hbar c}\xi^n(\vec{x});\ -id_0\xi^{3+n} = \frac{E}{\hbar c}\xi^{3+n}(\vec{x}), \qquad (3.156)\\
-id_0\eta^n &= \frac{E}{\hbar c}\eta^n(\vec{x});\ -id_0\eta^{3+n} = \frac{E}{\hbar c}\eta^{3+n}(\vec{x}).
\end{aligned}
$$

We then have:

$$
\begin{aligned}
-T_0^0 &= \frac{E}{\hbar c}\sum_{n=2}^{4}\Big[\frac{\mathbf{m}}{kq_1}\eta^{n\dagger}\eta^n + \frac{\mathbf{m}}{kq_2}\xi^{n\dagger}\xi^n + \frac{\mathbf{m}}{kq_3}\eta^{3+n\dagger}\eta^{3+n} + \frac{\mathbf{m}}{kq_4}\xi^{3+n\dagger}\xi^{3+n}\Big]\\
&= \frac{E}{\hbar c}(\frac{\mathrm{m}}{kq_1}S_L^d + \frac{\mathrm{m}}{kq_2}S_R^d + \frac{\mathrm{m}}{kq_3}S_L^u + \frac{\mathrm{m}}{kq_4}S_R^u)^0 = \frac{E}{\hbar c}\underline{\mathbf{J}}^0, \qquad (3.157)\\
S_L^d &= \sum_{n=2}^{4}\eta^{n\dagger}\eta^n;\ S_R^d = \sum_{n=2}^{4}\xi^{n\dagger}\xi^n;\ S_L^u = \sum_{n=2}^{4}\eta^{3+n\dagger}\eta^{3+n};\ S_R^u = \sum_{n=2}^{4}\xi^{3+n\dagger}\xi^{3+n},
\end{aligned}
$$

naming $\underline{\mathbf{J}}$ the weighted current with the relative weights m/kq_j. As in Chapter 2, this weighted current replaces the probability current of Chapter 1. The reason for the existence of a probability current in quantum mechanics is still the equivalence between inertial mass and gravitational mass, which implies:

$$0 = E + \iiint dvT_0^0; \quad \iiint \frac{\mathbf{J}^0}{\hbar c} dv = 1. \tag{3.158}$$

As for the lone electron or for the lepton wave we have two useful tensors of energy–momentum instead of only one. The second tensor is the V of Costa de Beauregard [51] that is obtained from the invariance under the translations of the Lagrangian density $\mathcal{L}_q^-$. This V reads:

$$kV = \sum_{n=2}^{4} \Big(\frac{\mathbf{m}}{q_1} T_L^n - \frac{\mathbf{m}}{q_2} T_R^n + \frac{\mathbf{m}}{q_3} T_L^{3+n} - \frac{\mathbf{m}}{q_4} T_R^{3+n} \Big). \tag{3.159}$$

The dynamics of the quarks comes from the variations of the energy–momentum tensor. The calculation is similar to that for the lepton wave. We have:

$$k\partial_\mu T^\mu = \sum_{n=2}^{4} \Big(\frac{\mathbf{m}}{q_1} \partial_\mu T_L^{n\mu} + \frac{\mathbf{m}}{q_2} \partial_\mu T_R^{n\mu} + \frac{\mathbf{m}}{q_3} \partial_\mu T_L^{3+n\mu} + \frac{\mathbf{m}}{q_4} \partial_\mu T_R^{3+n\mu} \Big). \tag{3.160}$$

For the first of these four terms we obtain:

$$\begin{aligned} &\partial_\mu T_L^{n\mu} = \partial_\mu T_{L\lambda}^{n\mu} \sigma^\lambda = \partial_\mu [i\eta^{n\dagger} \sigma^\mu \partial_\lambda \eta^n - g_{n\lambda}^1 \mathrm{D}_L^{n\mu}] \sigma^\lambda \\ &= [i(\nabla\eta^n)^\dagger \partial_\lambda \eta^n + i\eta^{n\dagger} \partial_\lambda (\nabla\eta^n) - (\partial_\mu g_{n\lambda}^1) \mathrm{D}_L^{n\mu} - g_{n\lambda}^1 \partial_\mu \mathrm{D}_L^{n\mu}] \sigma^\lambda. \end{aligned} \tag{3.161}$$

And we have with (3.105) and with (B.95) and (B.96):

$$\nabla\eta^n = -i(g_n^1 + q_1 \mathrm{v}_q)\eta^n, \tag{3.162}$$

$$\begin{aligned} \partial_\mu \mathrm{D}_L^{n\mu} &= (\partial_\mu \eta^{n\dagger}) \sigma^\mu \eta^n + \eta^{n\dagger} \sigma^\mu (\partial_\mu \eta^n) \\ &= i\eta^{n\dagger}(g_n^1 + q_1 \mathrm{v}_q)\eta^n - i\eta^{n\dagger}(g_n^1 + q_1 \mathrm{v}_q)\eta^n = 0. \end{aligned} \tag{3.163}$$

This gives:

$$i(\nabla\eta^n)^\dagger \partial_\lambda \eta^n + i\eta^{n\dagger} \partial_\lambda (\nabla\eta^n) \tag{3.164}$$

$$\begin{aligned} &= -\eta^{n\dagger}(g_n^1 + q_1 \mathrm{v}_q)\partial_\lambda \eta^n + \eta^{n\dagger}[(\partial_\lambda g_n^1)\eta^n + q_1(\partial_\lambda \mathrm{v}_q)\eta^n + (g_n^1 + q_1 \mathrm{v}_q)\partial_\lambda \eta^n] \\ &= (\partial_\lambda g_{n\mu}^1 + q_1 \partial_\lambda \mathrm{v}_{q\mu}) \mathrm{D}_L^{n\mu}. \end{aligned} \tag{3.165}$$

And we then get

$$-\partial_\mu T_L^{n\mu} = [(\partial_\mu g_{n\lambda}^1 - \partial_\lambda g_{n\mu}^1) + q_1 \partial_\lambda \mathrm{v}_{q\mu}] \mathrm{D}_L^{n\mu} \sigma^\lambda. \tag{3.166}$$

Similarly we next obtain:

$$-\partial_\mu T_R^{n\mu} = [(\partial_\mu g_{n\lambda}^2 - \partial_\lambda g_{n\mu}^2) + q_2 \partial_\lambda \mathrm{v}_{q\mu}] \mathrm{D}_R^{n\mu} \sigma^\lambda, \tag{3.167}$$

$$-\partial_\mu T_L^{3+n\mu} = [(\partial_\mu g_{n\lambda}^3 - \partial_\lambda g_{n\mu}^3) + q_3 \partial_\lambda \mathrm{v}_{q\mu}] \mathrm{D}_L^{3+n\mu} \sigma^\lambda, \tag{3.168}$$

$$-\partial_\mu T_R^{3+n\mu} = [(\partial_\mu g_{n\lambda}^4 - \partial_\lambda g_{n\mu}^4) + q_4 \partial_\lambda \mathrm{v}_{q\mu}] \mathrm{D}_R^{3+n\mu} \sigma^\lambda. \tag{3.169}$$

With

$$g^k = \frac{\mathbf{m}}{q_k}\sum_{n=2}^{4} g_n^k;\ \mathrm{J}_q = \sum_{n=2}^{7}(\mathrm{D}_R^n + \mathrm{D}_L^n), \tag{3.170}$$

we get:

$$-k\partial_\mu T^\mu = \begin{bmatrix} (\partial_\mu g_\lambda^1 - \partial_\lambda g_\mu^1)S_L^{d\mu} + (\partial_\mu g_\lambda^2 - \partial_\lambda g_\mu^2)S_R^{d\mu} \\ +(\partial_\mu g_\lambda^3 - \partial_\lambda g_\mu^3)S_L^{u\mu} + (\partial_\mu g_\lambda^4 - \partial_\lambda g_\mu^4)S_R^{u\mu} \end{bmatrix}\sigma^\lambda + \mathbf{m}(\partial_\lambda \mathrm{v}_{q\mu})\mathrm{J}_q^\mu \sigma^\lambda. \tag{3.171}$$

We are thus able to separate the forces acting on the wave into a part acting on the d quark and a part acting on the u quark, because the only term shared by the two parts is null. We indeed have:

$$\mathrm{J}_q^\mu = \rho_q \mathrm{v}_{q\mu};\ J_q^\mu \partial_\lambda \mathrm{v}_{q\mu} = \frac{\rho_q}{2}\partial_\lambda(\mathrm{v}_{q\mu}\mathrm{v}_q^\mu) = \frac{\rho_q}{2}\partial_\lambda(1) = 0. \tag{3.172}$$

We then have for the d quark:

$$-k\partial_\mu T_d^\mu = [(\partial_\mu g_\lambda^1 - \partial_\lambda g_\mu^1)S_L^{d\mu} + (\partial_\mu g_\lambda^2 - \partial_\lambda g_\mu^2)S_R^{d\mu}]\sigma^\lambda. \tag{3.173}$$

And similarly for the u:

$$-k\partial_\mu T_u^\mu = [(\partial_\mu g_\lambda^3 - \partial_\lambda g_\mu^3)S_L^{u\mu} + (\partial_\mu g_\lambda^4 - \partial_\lambda g_\mu^4)S_R^{u\mu}]\sigma^\lambda. \tag{3.174}$$

And we obtain:

$$\begin{aligned} g^1 &= \frac{\mathbf{m}}{q_1}(g_2^1 + g_3^1 + g_4^1) = \frac{\mathbf{m}}{q_1}\begin{bmatrix} -\frac{\mathrm{b}}{3} + 3\mathrm{w}_2^3 - 3\mathrm{h}_{L3}^{d3} + 3\mathrm{h}_{L1}^{d3} \\ -\frac{\mathrm{b}}{3} + 3\mathrm{w}_3^3 - 3\mathrm{h}_{L1}^{d3} + 3\mathrm{h}_{L2}^{d3} \\ -\frac{\mathrm{b}}{3} + 3\mathrm{w}_4^3 - 3\mathrm{h}_{L2}^{d3} + 3\mathrm{h}_{L3}^{d3} \end{bmatrix} \\ &= \frac{\mathbf{m}}{q_1}\Big[-\frac{g_1}{2}B + \frac{3g_2}{2}(\mathrm{D}_L^5 - \mathrm{D}_L^2 + \mathrm{D}_L^6 - \mathrm{D}_L^3 + \mathrm{D}_L^7 - \mathrm{D}_L^4)\Big] \\ &= \frac{\mathbf{m}}{q_1}\Big[-\frac{g_1}{2}B + \frac{3g_2}{2}(S_L^u - S_L^d)\Big]. \end{aligned} \tag{3.175}$$

Between the first and the second line the potentials of chromodynamics completely disappear. This implies that there are no strong forces for a wave of quarks that equally contains the three color states. And this is well known in nuclear physics where there are no stable states formed by three d quarks or three u quarks with color r, g and b.

This result is very important because this explains why we cannot place in the same wave the quarks composing a proton and a neutron. Above all this means that the proton or the neutron is a unique wave containing the three colored quarks. We now see that it is this wave of the proton or the neutron which has a quantized kinetic momentum, not a lone quark.

3.7 Quantization of the kinetic momentum

We may again use what we have shown in Chapter 2 for the lepton wave. First we have instead of (2.249):

$$kV_\lambda^\mu = \sum_{n=2}^{4} \Re\begin{bmatrix} -i(\frac{\mathbf{m}}{q_1}\eta^{n\dagger}\sigma^\mu d_\lambda \eta^n - \frac{\mathbf{m}}{q_2}\xi^{n\dagger}\widehat{\sigma}^\mu d_\lambda \xi^n) \\ -i(\frac{\mathbf{m}}{q_3}\eta^{3+n\dagger}\sigma^\mu d_\lambda \eta^{3+n} - \frac{\mathbf{m}}{q_4}\xi^{3+n\dagger}\widehat{\sigma}^\mu d_\lambda \xi^{3+n}) \end{bmatrix}. \tag{3.176}$$

We have twelve fields of spinors, six left and six right ones instead of lepton wave's four, but with very similar properties. Now we let:

$$\varphi_n = \eta^n;\ \varphi_{6+n} = \xi^n,\ n = 2, 3, \dots, 7. \tag{3.177}$$

Next, as in (2.257) and (2.259) :

$$\begin{aligned} \delta\varphi_a &= \phi_i^a \delta\omega^i \\ \eta^n + \delta\eta^n = \widehat{M}\eta^n&;\ \xi^n + \delta\xi^n = M\xi^n. \end{aligned} \tag{3.178}$$

We then obtain as in (2.262) to (2.265):

$$\begin{aligned} \phi_0^n = \frac{\eta^n}{2};\ \phi_1^n = -\sigma_1\frac{\eta^n}{2};\ \phi_2^n = -\sigma_2\frac{\eta^n}{2};\ \phi_3^n = -\sigma_3\frac{\eta^n}{2}, \\ \phi_4^n = i\sigma_1\frac{\eta^n}{2};\ \phi_5^n = i\sigma_2\frac{\eta^n}{2};\ \phi_6^n = i\sigma_3\frac{\eta^n}{2};\ \phi_7^n = -i\frac{\eta^n}{2}, \end{aligned} \tag{3.179}$$

$$\begin{aligned} \phi_0^{6+n} = \frac{\xi^n}{2};\ \phi_1^{6+n} = \sigma_1\frac{\xi^n}{2};\ \phi_2^{6+n} = \sigma_2\frac{\xi^n}{2};\ \phi_3^{6+n} = \sigma_3\frac{\xi^n}{2}, \\ \phi_4^{6+n} = i\sigma_1\frac{\xi^n}{2};\ \phi_5^{6+n} = i\sigma_2\frac{\xi^n}{2};\ \phi_6^{6+n} = i\sigma_3\frac{\xi^{6+n}}{2};\ \phi_7^{6+n} = i\frac{\xi^n}{2}. \end{aligned} \tag{3.180}$$

Hence we always have (2.266) to (2.270), without changes other than the replacement of the Lagrangian densities from the lepton wave by those from the waves of quarks. We then now have:

$$j_7^\mu = -\frac{\partial\mathcal{L}^-}{\partial(\partial_\mu\varphi_a)}\phi_7^a \tag{3.181}$$

$$\begin{aligned} j_7^\mu = \sum_{n=2}^{4}\Big[&i\frac{\mathbf{m}}{q_1}\eta^{n\dagger}\sigma^\mu\Big(\frac{-i}{2}\Big)\eta^n - i\frac{\mathbf{m}}{q_2}\xi^{n\dagger}\widehat{\sigma}^\mu\Big(\frac{i}{2}\Big)\xi^n \\ &+ i\frac{\mathbf{m}}{q_3}\eta^{3+n\dagger}\sigma^\mu\Big(\frac{-i}{2}\Big)\eta^{2+n} - i\frac{\mathbf{m}}{q_4}\xi^{3+n\dagger}\widehat{\sigma}^\mu\Big(\frac{i}{2}\Big)\xi^{3+n}\Big]. \end{aligned} \tag{3.182}$$

With (3.157) we then have:

$$j_7 = \frac{1}{2}\Big[\frac{\mathbf{m}}{kq_1}S_L^d + \frac{\mathbf{m}}{kq_2}S_R^d + \frac{\mathbf{m}}{kq_3}S_L^u + \frac{\mathbf{m}}{kq_4}S_R^u\Big] = \frac{1}{2}\underline{\mathbf{J}}_q. \tag{3.183}$$

And thus (3.158) gives:

$$\iiint dv\underline{\mathbf{J}}_q^0 = \hbar c;\ \iiint dv\frac{j_7^0}{c} = \frac{1}{2c}\iiint dv\underline{\mathbf{J}}_q^0 = \frac{\hbar}{2}. \tag{3.184}$$

This gives the quantization of kinetic momentum of the proton or the neutron. This satisfies all known properties of these two kinds of particles. The quantization of kinetic momentum is thus for the electrons and also for the protons and neutrons, a direct consequence of the wave equations of these particles and of the invariance under the extended group Cl_3^*.

This quantization of kinetic momentum, not of each of the quarks separately but of the proton and the neutron with their three colored quarks, has a very well-known experimental consequence: it is impossible to move a lone quark outside a proton or a neutron. In spite of the fact that they are made of several waves, only the protons and the neutrons have a possible individuality. This corresponds to the existence of a proper kinetic momentum that is always an odd integer multiple of $\hbar/2$. The only objects that we may see in an individual manner in any experiment of physics are those with a kinetic momentum multiple of $\hbar/2$. This kinetic momentum may be a multiple of $\hbar$ in the case of bosons. We may encounter $3\hbar/2$ for some hadrons or for states with five quarks. But the kinetic momentum cannot be smaller than $\hbar/2$ because all the objects that we can test in any physics experiment cannot have a kinetic momentum smaller than $\hbar/2$.

We recall that the quantization of kinetic momentum is at the origin of Heisenberg's inequalities (see [58]). These inequalities thus apply to any proton or neutron. The fact that quantum mechanics works similarly for an electron, a proton or a neutron gives the same "fundamental particle" character to these objects, and seems to question the quark model which is at the core of the Standard Model. Nevertheless it is the wave of a quark that is similar to the wave of an electron, not the wave of a proton or neutron. Once again, **this supports the Standard Model**.

3.7.1 Case of the lone proton or the lone neutron

The proton is made of two u quarks and one d quark. Since the color of the different nucleons does not add, the Standard Model says the proton (or the neutron) is color neutral. We may then suppose, for instance, that a proton is at a given instant made of a u_r quark, a u_g quark and a d_b quark. From our previous calculations the proton is then composed of only six non-null spinor waves: L^4, L^5, L^6 and R^4, R^5, R^6, all other spinor waves being exactly null. The J_q and $\underline{J}$ currents are thus the sum of only six spinor currents instead of the twelve possible currents:

$$J_q = D_L^4 + D_L^5 + D_L^6 + D_R^4 + D_R^5 + D_R^6. \tag{3.185}$$

The Lagrangian density of the proton hence comes from (3.136):

$$\begin{aligned}0 = \mathcal{L}^+ =& \frac{\mathbf{m}}{q_1}(-i\eta^{4\dagger}\sigma^\mu\partial_\mu\eta^4 + d_\mu^1\mathrm{D}_L^{4\mu}) + \frac{\mathbf{m}}{q_2}(-i\xi^{4\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^4 + d_\mu^2\mathrm{D}_R^{4\mu})\\ &+ \sum_{n=2}^{3}\begin{bmatrix}\frac{\mathbf{m}}{q_3}(-i\eta^{3+n\dagger}\sigma^\mu\partial_\mu\eta^{3+n} + d_\mu^3\mathrm{D}_L^{3+n\mu})\\ +\frac{\mathbf{m}}{q_4}(-i\xi^{3+n\dagger}\widehat{\sigma}^\mu\partial_\mu\xi^{3+n} + d_\mu^4\mathrm{D}_R^{3+n\mu})\end{bmatrix},\end{aligned} \tag{3.186}$$

where d^k is taken from (3.134), in which we replace n by 4 for the calculation of d^1 and d^2, and we consider only $n = 2$ and $n = 3$ for the calculation of d^3 and d^4. The same restrictions on the indices must be done in the calculation of the energy–momentum tensors and the kinetic momentum tensors, as well as in the calculation of the forces of inertia. The only part of the calculation that changes is in (3.175) where the square brackets include only one line instead of three. And the u quark uses two of the three lines of the square brackets. And it is the sum of the three lines that disappears, this comes from dimension 8 and not 9 of the group of chromodynamics. Strong interactions do not then disappear for a lone proton.

The case is similar for a neutron made of a u quark and two d quarks. There are also only six non-null spinor waves, for instance L^2, L^3, L^7 and R^2, R^3, R^7. The quantization of kinetic momentum applies to both proton and neutron – this is in accordance with particle physics. Protons and neutrons were discovered many years before the hypothesis that they are made of three quarks linked together by the forces of chromodynamics. The main problem of this hypothesis is the confinement of the quarks, the practical impossibility to bring a quark out of the bags that are mesons and baryons.

The previous calculation explains this confinement: the quantum of kinetic momentum exists for a proton made of three quarks or for a neutron, not for an isolated quark. This quantum of kinetic momentum exists also for a lone electron, a lone neutrino, or for an electron–neutrino pair. If a lone quark was able to get a quantum of kinetic momentum it should be possible to push this quark out of the bag, but there is no kinetic momentum lower than $\hbar/2$. Particles accelerators use electrons and protons which each have one quantum of kinetic momentum. The only objects that the collisions can produce also have a kinetic momentum $n\hbar/2$, where n is an integer that may be null if the object (a meson) contains two opposite $\hbar/2$ spins. Actually this quantum of kinetic momentum also explains another restriction: we never observe a left neutrino alone just as we never see a quark alone. We see a left neutrino only with a charged particle when and where they interact, or with a right neutrino, or with both a right neutrino and an electron or another particle with an electric charge. But the complete neutrino, which we may also call the magnetic monopole, has a quantum of kinetic momentum and can then be considered as an observable particle. And there is already some evidence of its being observed [36, 48, 49, 82].

3.8 Preference for left waves

Since the P transformation $P : M \mapsto \widehat{M}$ is an automorphism of Cl_3, the ring $\mathrm{End}(Cl_3) = Cl_{3,3}$ contains the subring $\mathrm{Diag}[\mathrm{End}(Cl_3)]$ of all:

$$\Psi_e = \begin{pmatrix} \psi_e & 0 \\ 0 & \psi_e \end{pmatrix} ; \ \psi_e = \begin{pmatrix} \phi_e & 0 \\ 0 & \widehat{\phi}_e \end{pmatrix} . \tag{3.187}$$

This subring may be considered as Cl_3 (because the P conjugation is the main automorphism), so Cl_3 is a subring of $\text{End}(Cl_3)$ and the operations of the Cl_3 ring are a particular case of the operations on $\text{End}(Cl_3)$. The result is an identification between the wave of first quantization ϕ_e and the wave of second quantization (an operator) Ψ_e. Now Cl_3 is also the set of the $\mathcal{P}_1 = \alpha + \mathbf{A} + \zeta\mathbf{i}$ of (B.96), which is the even subalgebra $Cl_{1,3}^+$ of $Cl_{1,3}$. We detailed this isomorphism in B.1.1 . With $\mathbf{A} = \vec{\mathbf{a}} + \mathbf{i}\vec{\mathbf{b}}$ we have $\mathcal{P}_1 = \alpha + \vec{\mathbf{a}} + \mathbf{i}\vec{\mathbf{b}} + \zeta\mathbf{i}$ whose self-adjoint part is $\alpha + \vec{\mathbf{a}}$. Quantum mechanics included space-time in this framework by setting $\mathrm{x} = \mathrm{x}^\mu\sigma_\mu$ (see (1.31), which gives

$$\det(\mathrm{x}) = (\mathrm{x}^0)^2 - (\mathrm{x}^1)^2 - (\mathrm{x}^2)^2 - (\mathrm{x}^3)^2. \tag{3.188}$$

This automatically introduces the $+\ -\ -\ -$ signature for space-time. It is the main reason for preferring $Cl_{1,3}$ to $Cl_{3,1}$. This other algebra could be still more important since $Cl_{3,1} = M_4(\mathbb{R})$ is the Majorana algebra. And $\text{End}(Cl_3) = M_8(\mathbb{R})$, each 8×8 real matrix comprising four 4×4 real matrices. Starting from the four γ_μ of (1.4) which generate $Cl_{1,3}$, the four $i\gamma_\mu$ generate $Cl_{3,1}$. The even subalgebra $Cl_{3,1}^+$ is thus the set of all

$$\widetilde{\mathcal{P}}_1 = \alpha - \mathbf{A} + \zeta\mathbf{i} = \alpha - \vec{\mathbf{a}} - \mathbf{i}\vec{\mathbf{b}} + \zeta\mathbf{i}. \tag{3.189}$$

Hence for space-time as the self-adjoint part of Cl_3, the passing from the $Cl_{1,3}^+$ version of Cl_3 to the $Cl_{3,1}^+$ version of the same Cl_3 induces a transformation from $\alpha + \vec{\mathbf{a}}$ to $\alpha - \vec{\mathbf{a}}$, which is the P (parity) transformation. It is the use of $Cl_{1,3}^+$ and the non-use of $Cl_{3,1}^+$ in Chapter 1 and Chapter 2 that fixes the preference for one of the two possible orientations of space, by the identification:

$$\phi_e \in Cl_3 = \psi_e \in Cl_{1,3}^+ = \Psi_e \in \text{Diag}[\text{End}(Cl_3)]. \tag{3.190}$$

This identification explains why second quantization may use all the results of first quantization in the electron case. The use of $Cl_{1,3}$ both for the electron wave and for space-time, as required by the determinant, leads to the use of $\nabla\widehat{\phi}_e$. And the left wave is the left column of $\widehat{\phi}_e$. Next there are two gauge invariances, the electric gauge generated by the 2-vector $\sigma_2\sigma_1 = -i\sigma_3$ and the chiral gauge generated by the 3-vector i. Under the electric gauge the left wave L_e rotates like $\widehat{\phi}_e$, the right wave R_e rotates like ϕ_e, and because $\widehat{i\sigma_3} = i\sigma_3$ then R_e rotates like L_e. Since $L(-i\sigma_3) = -iL$ then L_e rotates with opposite angles under the electric gauge and the chiral gauge. This results in the equalities of the coefficients of B for the left waves, seen in 2.2 in the lepton case and in 3.4 in the quark case. Next the value $\sin(\theta_W) = 1/2$ in (2.210) comes from the fact that A is the electric gauge potential. And this implies in (2.213) the suppression of the only term $qA\overline{L}^8$. The term $qA\overline{R}^8$ is not suppressed. This is the origin of the "maximal parity violation" in weak interactions.

Chapter 4

Gravitation

The space-time manifold of general relativity is a submanifold in Cl_3^*. The connection of this manifold is calculated both from the quantum wave (gravitation) and from the invariance group (inertia). Equations of the gravitational field are as in general relativity an equivalence between inertia and gravitation. The i which defines the orientation of space belongs both to the invariance group and to the gauge group. We generalize the form-invariant derivative. This derivative simplifies the weak interactions part for quark waves. We study the double link between wave equations in the usual form and form-invariant equations, its consequences on the conservation of currents, and the homothety ratio. We show the compatibility between gravitation and our results for the energy–momentum and kinetic momentum tensor densities. We revisit the link between the Pauli principle and gravitation. Instead of propagating in a linear configuration space, the fermion wave propagates in the space-time manifold. The whole space-time accounts for the possible resolution of the EPR problem. The duality of the Lie group versus Lie algebra explains the arrow of time and the expansion of the universe.

The space-time manifold of general relativity has naturally been thought of as a Riemannian manifold based on properties of the space-time metric. With our choice (1.4) of matrices we have:

$$\mathbf{x} := \mathrm{x}^\mu \gamma_\mu = \begin{pmatrix} 0 & \mathrm{x} \\ \widehat{\mathrm{x}} & 0 \end{pmatrix};\ \mathrm{x} = \mathrm{x}^\mu \sigma_\mu. \tag{4.1}$$

The similitude R defined by the dilator M satisfies with (1.34):

$$\mathbf{x}' = \begin{pmatrix} 0 & \mathrm{x}' \\ \widehat{\mathrm{x}}' & 0 \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{x} \\ \widehat{\mathrm{x}} & 0 \end{pmatrix} \begin{pmatrix} \overline{M} & 0 \\ 0 & M^\dagger \end{pmatrix}, \tag{4.2}$$

$$\mathbf{x}' = N\mathbf{x}\widetilde{N};\ N := \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix};\ \widetilde{N} = \begin{pmatrix} \overline{M} & 0 \\ 0 & M^\dagger \end{pmatrix}. \tag{4.3}$$

The electron, in the tangent space-time at each point of the space-time manifold, has a wave following the improved Dirac equation described in Chapter 1. The set of N is the even subalgebra $Cl^+_{1,3}$, isomorphic to Cl_3. The Dirac theory restricts M to $SL(2,\mathbb{C})$, which implies:

$$\det(M) = 1;\ M^{-1} = \overline{M}, \tag{4.4}$$

$$\nabla\widehat{\phi} = \sigma^\nu\partial_\nu\widehat{\phi} = \overline{M}\nabla'\widehat{M}\widehat{\phi} = M^{-1}\sigma^\mu\partial'_\mu M^{\dagger-1}\partial'_\mu\widehat{\phi} = M^{-1}\sigma^\mu M^{\dagger-1}R^\nu_\mu\partial_\nu\widehat{\phi}$$

$$\sigma^\nu = M^{-1}\sigma^\mu R^\nu_\mu (M^\dagger)^{-1};\ \sigma^\mu R^\nu_\mu = M\sigma^\nu M^\dagger;\ \gamma^\mu R^\nu_\mu = N\gamma^\nu\widetilde{N}. \tag{4.5}$$

This calculation is valid as soon as M is a fixed term. But the last relations (one relation for each value of ν), are carelessly used even in the case of a variable M. Moreover the N matrix is generally supposed to satisfy:

$$N = \sum_{a<b} e^{\theta^{ab}\boldsymbol{\sigma}_{ab}};\ \boldsymbol{\sigma}_{ab} := \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a),\ \theta^{ab} \in \mathbb{R}. \tag{4.6}$$

But in the previous calculations the similitude R is not the dilator M. Even if M belongs to $SL(2,\mathbb{C})$ the Lorentz transformation R is not the Pauli matrix M. And unhappily (4.6) may be false: with $M = -1 + \sigma_1 + i\sigma_2$ we obtain:

$$M = -\exp[-(\sigma_1 + i\sigma_2)] = e^{i\pi - \sigma_0\sigma_1 + \sigma_1\sigma_3}, \tag{4.7}$$

$$N = e^{\pi\boldsymbol{\sigma}_{01}\boldsymbol{\sigma}_{23} + \boldsymbol{\sigma}_{01} + \boldsymbol{\sigma}_{13}} = e^{\pi(1+2k)\boldsymbol{\sigma}_{01}\boldsymbol{\sigma}_{23} + \boldsymbol{\sigma}_{01} + \boldsymbol{\sigma}_{13}},\ k \in \mathbb{Z}. \tag{4.8}$$

Thus any calculation based on (4.6) may be false: we must adopt another approach, which we now develop.

The quantum wave studied in previous chapters introduces a major change with the inclusion (1.31) of space-time into Cl_3, first realized by Pauli nearly a century ago. This inclusion allows us to obtain the metric by:

$$||\mathrm{x}||^2 = \det(\mathrm{x}) = \mathrm{x}\widehat{\mathrm{x}} = \mathrm{x}P(\mathrm{x}). \tag{4.9}$$

where P is the parity transformation. This is indeed very different from the approach of Riemannian geometry: first the norm $||\mathrm{x}||$ is not a true norm since $\det(\mathrm{x})$ may be positive, zero or negative. Next a determinant is not a symmetric bilinear form but an antisymmetric one. Moreover the parity transformation is directly associated to geometry by (4.9). We encountered in the first chapter another important relation (1.279):

$$D_\mathrm{x} : X \mapsto \mathrm{x} = \phi X \phi^\dagger$$

defining the general element X of what we termed "invariant space-time". Since $\det(AB) = \det(A)\det(B)$ and since the M_ϕ element of $SL(2,\mathbb{C})$ was defined in (1.157) as $\phi = \sqrt{\rho}e^{i\beta/2}M_\phi$, we have:

$$\det(\mathrm{x}) = \rho \det(X). \tag{4.10}$$

We saw that for each important solution used in the Dirac theory, $\rho = \rho(\mathrm{x})$ is nonzero everywhere. And since no observer can travel on the light cone, x satisfies $\det(\mathrm{x}) \neq 0$, we will then make the hypothesis that $\det(X) \neq 0$. This means that X belongs to the Lie group Cl_3^* and the set of X is the self-adjoint part of Cl_3^*, satisfying $X = X^\dagger$. The determinant of a product is the product of determinants; thus the relation between X and x implies that x also has a nonzero determinant: this means that the set of x, which is the space-time manifold, is itself the self-adjoint part of the Lie group Cl_3^*. Whitney's theorem indicates that any 4-dimensional manifold may be included in $\mathbb{R}^8$, and Cl_3 is precisely 8-dimensional on $\mathbb{R}$. Therefore space-time geometry does not need Cl_3 itself (isomorphic to $\mathbb{R}^8$ as a manifold) but geometry needs the Lie group Cl_3^* itself, which is an 8-dimensional separate manifold having Cl_3 as its Lie algebra. Since it is a Lie group, each point is locally identical to the unity point $\mathrm{x} = 1$. This unity point, as any one, means an event: "I am here, now". At this point-event, a tangent space-time exists, where this event "I am here, now" becomes the zero point of the Lie algebra Cl_3 of the Cl_3^* Lie group. Since distances are given by the determinant and since this determinant is not null, by definition of Cl_3^*, the light cone of each point-event is only a subset of the tangent space-time at the considered point-event. This tangent space-time must be distinguished from space-time itself, because the tangent space-time is itself a flat space.

We saw in 1.1.2 and 1.2 that the invariance group $\mathcal{G}$ generalizing to the relativistic case the invariance group $SU(2)$ of nonrelativistic quantum physics for the particles with spin 1/2, may be, without any difficulty, extended to the $GL(2,\mathbb{C})$ group, which is the group of all 2×2 complex invertible matrices. This group is isomorphic to the Cl_3^* group, consisting of all invertible elements in the Cl_3 algebra. This algebra contains the group of its invertible elements, and moreover it is the Lie algebra of this Lie group. In the previous chapters, we studied the first consequences of this generalization of relativistic invariance. The difference between the dilator M and the similitude R generated by a dilator is the same as in the particular case of Lorentz transformation: the dilator group is a 8-dimensional Lie group while the similitude group is only a 7-dimensional group. The kernel of the homomorphism $f : M \mapsto R$ is the 1-dimensional group made of $M = e^{ix}$, where x is any real number. The f function cannot be invertible: no way exists to define M from R. It is the true reason explaining why Cl_3 is the most important linear space and why Cl_3^* is the most important invariance group used in this Chapter. It is so because it is impossible otherwise: the isomorphism between $GL(n,\mathbb{C})$ and Cl_p^*, as Lie groups, exists only if $n = 2$ and $p = 3$.

4.1 Differential geometry

4.1.1 Gravitation from the quantum wave

All differential operators that we used in previous chapters are built from the $\nabla = \sigma^\mu \partial_\mu$ operator of the Cl_3 algebra. The operator $\boldsymbol{\partial} = \gamma^\mu \partial_\mu$ used by Hestenes depends on ∇ since we have (see B.1.2):

$$\boldsymbol{\partial} = \begin{pmatrix} 0 & \nabla \end{pmatrix}. \tag{4.11}$$

Thus Hestenes' space-time algebra is only a possible help in calculations, but has no real necessity: we hence use here only the Cl_3 algebra. We go from the operator ∇ that operates in the neighborhood of the event $\mathrm{x} = \mathrm{x}^\mu \sigma_\mu$ to the operator ∇' that operates in the neighborhood of the event $\mathrm{x}' = \mathrm{x}'^\mu \sigma_\mu$ by:

$$\mathrm{x}' = R(\mathrm{x}) = M\mathrm{x}M^\dagger;\ \nabla = \overline{M}\nabla'\widehat{M};\ \nabla' = \sigma^\mu \partial'_\mu. \tag{4.12}$$

We recall that σ_μ is exactly the same in writing either x or x′, and likewise for ∇ and ∇'.[1] We explained in 1.8 how the ϕ wave of the electron defines a similitude:

$$D_{\mathrm{x}} : X = X^\mu \sigma_\mu \mapsto \mathrm{x} = \phi X \phi^\dagger = \phi X^\nu \sigma_\nu \phi^\dagger = X^\nu \mathrm{D}_\nu, \tag{4.13}$$

$$\mathrm{D}_\nu = \mathrm{D}^\mu_\nu \sigma_\mu = \phi \sigma_\nu \phi^\dagger. \tag{4.14}$$

So the ϕ function enables an immersion of the space-time manifold into the 8-dimensional manifold, seen from our immersed manifold. The function ϕ is all that may be seen from our manifold.

We saw in 1.3.2 that the D_ν vectors form an orthogonal basis of space-time (but not an orthonormal basis), at each point. To this variable basis is thus associated an affine connection. This also allows us to use Cartan's $(\mathrm{D}_0, \mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3)$ mobile basis. These vectors are calculated in A.4.2. We recall that we have:

$$\rho e^{i\beta} = \phi\overline{\phi} = \det(\phi);\ \rho e^{-i\beta} = \widehat{\phi}\widetilde{\phi}, \tag{4.15}$$

$$\mathrm{D}_\mu \cdot \mathrm{D}_\nu = 0,\ \mu \neq \nu, \tag{4.16}$$

$$\rho^2 = \mathrm{D}_0 \cdot \mathrm{D}_0 = -\mathrm{D}_1 \cdot \mathrm{D}_1 = -\mathrm{D}_2 \cdot \mathrm{D}_2 = -\mathrm{D}_3 \cdot \mathrm{D}_3. \tag{4.17}$$

This connection was first studied in [22]. We let:

$$\boldsymbol{\partial}_\nu = \frac{\partial}{\partial X^\nu} = \mathrm{D}^\mu_\nu \partial_\mu;\ d\mathrm{x} = dX^\nu \mathrm{D}_\nu, \tag{4.18}$$
$$d\mathrm{D}_\mu = \Gamma^\beta_{\mu\nu} dX^\nu \mathrm{D}_\beta.$$

1. The fixedness of the σ^μ comes from the fact that the four matrices $(1 \pm \sigma_3)/2$ and $(\sigma_1 \pm i\sigma_2)/2$ constitute the canonical basis of $M_2(\mathbb{C})$ and are thus intrinsic in the $GL(2, \mathbb{C})$ group.

If $\rho \neq 0$ this gives:

$$d\mathrm{x} = d\mathrm{x}^{\mu}\sigma_{\mu} = \mathrm{D}^{\mu}_{\nu}\sigma_{\mu}dX^{\nu} = \mathrm{D}_{\nu}dX^{\nu},$$
$$\mathrm{D}_{\nu} = \phi\sigma_{\nu}\phi^{\dagger} = \mathrm{D}^{\mu}_{\nu}\sigma_{\mu} \ ; \quad \sigma_{\mu} = (\mathrm{D}^{-1})^{\beta}_{\mu}\mathrm{D}_{\beta}. \tag{4.19}$$

Now we use the similitude $\overline{\mathrm{D}}$ such that

$$\overline{\mathrm{D}}(\mathrm{x}) = \overline{\phi}\mathrm{x}\widehat{\phi}. \tag{4.20}$$

We have

$$\mathrm{D}\circ\overline{\mathrm{D}}(\mathrm{x}) = \mathrm{D}[\overline{\mathrm{D}}(\mathrm{x})] = \phi\overline{\phi}\mathrm{x}\widehat{\phi}\phi^{\dagger} = \rho e^{i\beta}\mathrm{x}\rho e^{-i\beta} = \rho^2\mathrm{x},$$
$$\mathrm{D}\circ(\rho^{-2}\overline{\mathrm{D}})(\mathrm{x}) = \mathrm{x},$$
$$\mathrm{D}^{-1}(\mathrm{x}) = \rho^{-2}\overline{\mathrm{D}}(\mathrm{x}). \tag{4.21}$$

And we get:

$$d\mathrm{D}_{\mu} = \boldsymbol{\partial}_{\nu}(\mathrm{D}_{\mu})dX^{\nu} = \boldsymbol{\partial}_{\nu}(\mathrm{D}^{\xi}_{\mu}\sigma_{\xi})dX^{\nu} = \boldsymbol{\partial}_{\nu}(\mathrm{D}^{\xi}_{\mu})\sigma_{\xi}dX^{\nu}$$
$$= \boldsymbol{\partial}_{\nu}(\mathrm{D}^{\xi}_{\mu})(\mathrm{D}^{-1})^{\beta}_{\xi}\mathrm{D}_{\beta}dX^{\nu} = \Gamma^{\beta}_{\mu\nu}\mathrm{D}_{\beta}dX^{\nu}. \tag{4.22}$$

Therefore the connection coefficients are

$$\Gamma^{\beta}_{\mu\nu} = \boldsymbol{\partial}_{\nu}(\mathrm{D}^{\xi}_{\mu})(\mathrm{D}^{-1})^{\beta}_{\xi} \ ; \quad \boldsymbol{\partial}_{\nu} = \mathrm{D}^{\tau}_{\nu}\partial_{\tau}. \tag{4.23}$$

By using the $\overline{\mathrm{D}}$ similitude we get

$$\Gamma^{\beta}_{\mu\nu} = \rho^{-2}\boldsymbol{\partial}_{\nu}(\mathrm{D}^{\xi}_{\mu})\overline{\mathrm{D}}^{\beta}_{\xi} \ ; \quad \boldsymbol{\partial}_{\nu} = \mathrm{D}^{\tau}_{\nu}\partial_{\tau}. \tag{4.24}$$

Since $\overline{\mathrm{D}}^0_0 = \mathrm{D}^0_0$ and $\overline{\mathrm{D}}^0_j = -\mathrm{D}^j_0$ we have:

$$\Gamma^0_{0\nu} = \Gamma^1_{1\nu} = \Gamma^2_{2\nu} = \Gamma^3_{3\nu} = \boldsymbol{\partial}_{\nu}[\ln(\rho)] = \mathrm{D}^{\mu}_{\nu}\partial_{\mu}[\ln(\rho)]. \tag{4.25}$$

Since $\overline{\mathrm{D}}^j_0 = -\mathrm{D}^0_j$ and $\overline{\mathrm{D}}^k_j = \mathrm{D}^j_k$ we have:

$$\Gamma^j_{0\nu} = \Gamma^0_{j\nu} \ , \quad j = 1,\ 2,\ 3, \tag{4.26}$$
$$\Gamma^j_{k\nu} = -\Gamma^k_{j\nu} \ , \quad j = 1,\ 2,\ 3 \ , \quad k = 1,\ 2,\ 3, \quad k \neq j. \tag{4.27}$$

A complete calculation of the connection needs the following quantities:

$$S_k = \phi\sigma_k\overline{\phi}, \tag{4.28}$$
$$\mathcal{S}_{(k)} + i\mathcal{S}'_{(k)} = \frac{\nabla S^{\dagger}_k}{\det(\phi)^{\dagger}}, \tag{4.29}$$
$$\mathcal{A}_{(k)} + i\mathcal{A}'_{(k)} = \frac{AS^{\dagger}_k}{\det(\phi)^{\dagger}}, \tag{4.30}$$
$$\tau = \frac{1}{2}[(\nabla\widehat{\phi})\phi^{\dagger} - \sigma^{\mu}\widehat{\phi}\partial_{\mu}\phi^{\dagger}], \tag{4.31}$$
$$\mathcal{T} + i\mathcal{T}' = \frac{\tau}{\det(\phi)^{\dagger}}. \tag{4.32}$$

The tensor τ is Durand's spin density [16, 51]. Using our improved wave equation of the electron, we obtain in D.4 :

$$\Gamma^0_{1\mu} = \mathrm{D}_\mu \cdot [\mathcal{S}_{(1)} - 2q\mathcal{A}_{(2)}] + 2m\rho\delta^2_\mu, \tag{4.33}$$
$$\Gamma^0_{2\mu} = \mathrm{D}_\mu \cdot [\mathcal{S}_{(2)} + 2q\mathcal{A}_{(1)}] - 2m\rho\delta^1_\mu, \tag{4.34}$$
$$\Gamma^0_{3\mu} = \mathrm{D}_\mu \cdot \mathcal{S}_{(3)}, \tag{4.35}$$
$$\Gamma^2_{3\mu} = -\mathrm{D}_\mu \cdot [\mathcal{S}'_{(1)} + 2q\mathcal{A}'_{(2)}] - 2d\rho\delta^1_\mu, \tag{4.36}$$
$$\Gamma^3_{1\mu} = -\mathrm{D}_\mu \cdot [\mathcal{S}'_{(2)} + 2q\mathcal{A}'_{(1)}] + 2d\rho\delta^2_\mu, \tag{4.37}$$
$$\Gamma^1_{2\mu} = -\mathrm{D}_\mu \cdot [\mathcal{S}'_{(3)} + 2qA] - 2m\rho\delta^0_\mu + 2d\rho\delta^3_\mu, \tag{4.38}$$
$$\Gamma^0_{0\mu} = \mathrm{D}_\mu \cdot [-2\mathcal{T} + 2q\mathcal{A}'_{(3)}], \tag{4.39}$$

where $\delta^0_0 = 1$, $\delta^j_j = -1, j = 1,2,3$ and $\delta^\nu_\mu = 0, \mu \neq \nu$. The particular role of the index 3 in the Dirac equation of the electron is still very visible in these relations. For the second or the third generation it is enough to make a circular permutation of indices. So a particular index, 1 or 2, is thus also visible. The connection is not torsion-free, and the proper mass is linked to this torsion: this is the reason to think of this connection as yielding gravitation. Moreover, the mass term $m\rho$, and thus also Christoffel's symbols, have the physical dimension L^{-1} of a radius of curvature. We may thus consider that **the link between mass-energy and geometry is not made with the curvature tensor, but directly with the affine connection and the torsion of the space-time manifold**. This is a generalization of Einstein's attempt at a space-time manifold without curvature and with torsion to account for both gravitation and electromagnetism [65].

With the plane wave we obtained in 1.5.3:

$$\phi = \phi_0 e^{-\varphi\sigma_{12}} \ ; \ \ \varphi = m_g \mathrm{v}_\mu x^\mu \ ; \ \ \mathrm{v} = \sigma^\mu \mathrm{v}_\mu, \ m_g = \sqrt{\mathbf{lr}}. \tag{4.40}$$

where the reduced velocity v and ϕ_0 are fixed terms. We obtain:

$$\phi = \sigma^\mu \partial_\mu (\widehat{\phi}_0 e^{-\varphi\sigma_{12}}) = -m_g \mathrm{v}\widehat{\phi}\sigma_{12}. \tag{4.41}$$

This gives

$$\phi = e^{i\beta}\mathrm{v}\widehat{\phi}\frac{\widehat{\mathbf{m}}}{m_g}, \tag{4.42}$$

which implies that

$$\phi = e^{i\beta}\mathrm{v}\Big(e^{-i\beta}\widehat{\mathrm{v}}\phi\frac{\mathbf{m}}{m_g}\Big)\frac{\widehat{\mathbf{m}}}{m_g} = \mathrm{v}\widehat{\mathrm{v}}\phi. \tag{4.43}$$

Therefore if ϕ_0 is invertible we must take

$$1 = \mathrm{v}\widehat{\mathrm{v}} = \mathrm{v}\cdot\mathrm{v} = \mathrm{v}_0^2 - \vec{\mathrm{v}}^{\,2}, \tag{4.44}$$
$$\mathrm{v}_0^2 = 1 + \vec{\mathrm{v}}^{\,2} \ ; \ \ \mathrm{v}_0 = \pm\sqrt{1 + \vec{\mathrm{v}}^{\,2}}. \tag{4.45}$$

which is the relativistic relation for the velocity of the particle. We also get:

$$\rho e^{i\beta} = \det(\phi) = \det(\phi_0)\det(e^{i\beta}) = \det(\phi_0). \tag{4.46}$$

Therefore ρ and β are fixed. It is the same for

$$\mathrm{D}_0 = \phi_0\phi_0^\dagger \ ; \quad \mathrm{D}_3 = \phi_0\sigma_3\phi_0^\dagger. \tag{4.47}$$

D_1 and D_2, on the contrary, are variable. We let

$$d_1 = \phi_0\sigma_1\phi_0^\dagger \ ; \quad d_2 = \phi_0\sigma_2\phi_0^\dagger, \tag{4.48}$$

which gives:

$$\begin{aligned}\mathrm{D}_1 &= \cos(2\varphi)d_1 + \sin(2\varphi)d_2,\\ \mathrm{D}_2 &= -\sin(2\varphi)d_1 + \cos(2\varphi)d_2.\end{aligned} \tag{4.49}$$

As D_0 and D_3 are fixed we get:

$$\boldsymbol{\partial}_\nu(\mathrm{D}_0^\xi) = \boldsymbol{\partial}_\nu(\mathrm{D}_3^\xi) = 0 \tag{4.50}$$

$$\Gamma_{0\nu}^\beta = \Gamma_{3\nu}^\beta = 0. \tag{4.51}$$

With D_1 and D_2 we obtain:

$$\begin{aligned}\partial_\tau(\mathrm{D}_1^\xi) &= \partial_\tau[\cos(2\varphi)d_1^\xi + \sin(2\varphi)d_2^\xi] = 2m_g\mathrm{v}_\tau\mathrm{D}_2^\xi,\\ \partial_\tau(\mathrm{D}_2^\xi) &= \partial_\tau[-\sin(2\varphi)d_1^\xi + \cos(2\varphi)d_2^\xi] = -2m_g\mathrm{v}_\tau\mathrm{D}_1^\xi,\end{aligned}$$

$$\boldsymbol{\partial}_\nu(\mathrm{D}_1^\xi) = \mathrm{D}_\nu^\tau\partial_\tau(\mathrm{D}_1^\xi) = 2m_g\mathrm{D}_\nu^\tau\mathrm{v}_\tau\mathrm{D}_2^\xi = 2m_g(\mathrm{D}_\nu\cdot\mathrm{v})\mathrm{D}_2^\xi, \tag{4.52}$$

$$\boldsymbol{\partial}_\nu(\mathrm{D}_2^\xi) = \mathrm{D}_\nu^\tau\partial_\tau(\mathrm{D}_2^\xi) = -2m - g\mathrm{D}_\nu^\tau\mathrm{v}_\tau\mathrm{D}_1^\xi = -2m_g(\mathrm{D}_\nu\cdot\mathrm{v})\mathrm{D}_1^\xi. \tag{4.53}$$

Next we get:

$$\mathrm{D}_\nu\cdot\mathrm{v} = \frac{1}{\rho}\mathrm{D}_\nu\cdot\mathrm{D}_0 = \rho\delta_\nu^0. \tag{4.54}$$

Therefore we have:

$$\Gamma_{11}^\beta = \Gamma_{12}^\beta = \Gamma_{13}^\beta = \Gamma_{21}^\beta = \Gamma_{22}^\beta = \Gamma_{23}^\beta = 0. \tag{4.55}$$

And we get:

$$\Gamma_{10}^\beta = \frac{2m_g}{\rho}\mathrm{D}_2^\xi\overline{\mathrm{D}}_\xi^\beta \ ; \quad \Gamma_{20}^\beta = -\frac{2m_g}{\rho}\mathrm{D}_1^\xi\overline{\mathrm{D}}_\xi^\beta, \tag{4.56}$$

which gives:

$$\begin{aligned}\Gamma_{10}^2 &= \frac{2m_g}{\rho}\mathrm{D}_2^\xi\overline{\mathrm{D}}_\xi^2 = \frac{2m_g}{\rho}(\mathrm{D}_2^0\overline{\mathrm{D}}_0^2 + \mathrm{D}_2^1\overline{\mathrm{D}}_1^2 + \mathrm{D}_2^2\overline{\mathrm{D}}_2^2 + \mathrm{D}_2^3\overline{\mathrm{D}}_3^2)\\ &= \frac{2m_g}{\rho}(-\mathrm{D}_2^0\mathrm{D}_2^0 + \mathrm{D}_2^1\mathrm{D}_2^1 + \mathrm{D}_2^2\mathrm{D}_2^2 + \mathrm{D}_2^3\mathrm{D}_2^3)\\ &= \frac{2m_g}{\rho}(-\mathrm{D}_2\cdot\mathrm{D}_2) = 2m_g\rho.\end{aligned} \tag{4.57}$$

We also have:

$$\Gamma^0_{10} = \frac{2m_g}{\rho}(\mathrm{D}_2 \cdot \mathrm{D}_0) = 0,$$
$$\Gamma^3_{10} = \frac{2m_g}{\rho}(-\mathrm{D}_2 \cdot \mathrm{D}_3) = 0,$$
$$\Gamma^1_{10} = \frac{2m_g}{\rho}(-\mathrm{D}_2 \cdot \mathrm{D}_1) = 0. \qquad (4.58)$$

Similarly for the Γ^β_{20} we get

$$\Gamma^1_{20} = -2m_g\rho\ ; \quad \Gamma^0_{20} = \Gamma^2_{20} = \Gamma^3_{20} = 0. \qquad (4.59)$$

To resume, among the 64 $\Gamma^\beta_{\mu\nu}$ terms, 62 terms are zero. Two terms are not zero:

$$\Gamma^2_{10} = -\Gamma^1_{20} = 2m_g\rho. \qquad (4.60)$$

Therefore the torsion has two fixed components:

$$\frac{1}{2}(\Gamma^2_{10} - \Gamma^2_{01}) = m_g\rho, \qquad (4.61)$$
$$\frac{1}{2}(\Gamma^1_{20} - \Gamma^1_{02}) = -m_g\rho. \qquad (4.62)$$

As the nonvanishing $\Gamma^\beta_{\mu\nu}$ terms are fixed, the curvature tensor cancels out. We thus see that, for the improved equation, the manifold linked to a plane wave is without curvature but with a fixed torsion, and the mass term is proportional to this torsion.

4.1.2 Inertia from the invariance group

We now have four kinds of spinors which vary in four different transformations: the $Cl_3^* = GL(2,\mathbb{C})$ group has four kinds of representations. So in addition to the invariance of what do not change such as the Lagrangian density, we get no less than six kinds of variance: the contravariance of vectors transforming like x, the covariance of vectors transforming like ∇, and four kinds of spinors that we encountered in previous chapters:

$$\mathrm{x}' = M\mathrm{x}M^\dagger;\ \nabla = \overline{M}\nabla'\widehat{M} = \overline{M}\sigma^\mu\widehat{M}\partial'_\mu, \qquad (4.63)$$
$$R'^n = MR^n;\ \widehat{L}'^n = \widehat{M}\widehat{L}^n,\ n = 1,2,3,4, \qquad (4.64)$$
$$R'^{4+n} = R^{4+n}\widetilde{M};\ \widehat{L}'^{4+n} = \widehat{L}^{4+n}\overline{M},\ n = 1,2,3,4. \qquad (4.65)$$

Differential geometry studies what happens in the neighborhood of a given point-event. This is equivalent to considering in the neighborhood of x a dilator M which differs from unity only by an infinitesimal. We thus let:

$$M = 1 + \mathrm{dx}^\mu(a^0_\mu + a^1_\mu\sigma_1 + a^2_\mu\sigma_2 + a^3_\mu\sigma_3 + a^4_\mu i\sigma_1 + a^5_\mu i\sigma_2 + a^6_\mu i\sigma_3 + a^7_\mu i), \qquad (4.66)$$

where the a_μ^n, for $\mu = 1, 2, 3, 4$ and $n = 0, 1, \ldots, 7$ are 32 smooth enough real functions of x, and dx^μ are increments of x at this point-event in the relevant local basis. This gives:

$$\begin{aligned}
M^\dagger &= 1 + \mathrm{dx}^\mu(a_\mu^0 + a_\mu^1\sigma_1 + a_\mu^2\sigma_2 + a_\mu^3\sigma_3 - a_\mu^4 i\sigma_1 - a_\mu^5 i\sigma_2 - a_\mu^6 i\sigma_3 - a_\mu^7 i), \\
\widehat{M} &= 1 + \mathrm{dx}^\mu(a_\mu^0 - a_\mu^1\sigma_1 - a_\mu^2\sigma_2 - a_\mu^3\sigma_3 + a_\mu^4 i\sigma_1 + a_\mu^5 i\sigma_2 + a_\mu^6 i\sigma_3 - a_\mu^7 i), \\
\overline{M} &= 1 + \mathrm{dx}^\mu(a_\mu^0 - a_\mu^1\sigma_1 - a_\mu^2\sigma_2 - a_\mu^3\sigma_3 - a_\mu^4 i\sigma_1 - a_\mu^5 i\sigma_2 - a_\mu^6 i\sigma_3 + a_\mu^7 i).
\end{aligned} \tag{4.67}$$

We also have:

$$M\overline{M} = \det(M) = 1 + 2\mathrm{dx}^\mu(a_\mu^0 + ia_\mu^7), \tag{4.68}$$

$$\det(M^{-1}) = 1 - 2\mathrm{dx}^\mu(a_\mu^0 + ia_\mu^7), \tag{4.69}$$

$$\overline{M}^{-1} = M\det(M^{-1}), \tag{4.70}$$

$$= 1 + \mathrm{dx}^\mu(-a_\mu^0 + a_\mu^1\sigma_1 + a_\mu^2\sigma_2 + a_\mu^3\sigma_3 + a_\mu^4 i\sigma_1 + a_\mu^5 i\sigma_2 + a_\mu^6 i\sigma_3 - a_\mu^7 i),$$

$$\widehat{M}^{-1} = (\overline{M}^{-1})^\dagger \tag{4.71}$$

$$= 1 + \mathrm{dx}^\mu(-a_\mu^0 + a_\mu^1\sigma_1 + a_\mu^2\sigma_2 + a_\mu^3\sigma_3 - a_\mu^4 i\sigma_1 - a_\mu^5 i\sigma_2 - a_\mu^6 i\sigma_3 + a_\mu^7 i).$$

The similitude R defined by M which changes x into x′, such that $\mathrm{x}' = R(\mathrm{x}) + \mathrm{a} = M\mathrm{x}M^\dagger + \mathrm{a}$, where a is the vector $\mathrm{a} = \mathrm{a}^\mu\sigma_\mu$ of a translation, gives the following:

$$\mathrm{x}'^0 = \mathrm{x}^0 + \mathrm{dx}^0 + 2(a_\mu^0\mathrm{x}^0 + a_\mu^1\mathrm{x}^1 + a_\mu^2\mathrm{x}^2 + a_\mu^3\mathrm{x}^3)\mathrm{dx}^\mu, \tag{4.72}$$

$$\mathrm{x}'^1 = \mathrm{x}^1 + \mathrm{dx}^1 + 2(a_\mu^1\mathrm{x}^0 + a_\mu^0\mathrm{x}^1 + a_\mu^6\mathrm{x}^2 - a_\mu^5\mathrm{x}^3)\mathrm{dx}^\mu, \tag{4.73}$$

$$\mathrm{x}'^2 = \mathrm{x}^2 + \mathrm{dx}^2 + 2(a_\mu^2\mathrm{x}^0 - a_\mu^6\mathrm{x}^1 + a_\mu^0\mathrm{x}^2 + a_\mu^4\mathrm{x}^3)\mathrm{dx}^\mu, \tag{4.74}$$

$$\mathrm{x}'^3 = \mathrm{x}^3 + \mathrm{dx}^3 + 2(a_\mu^3\mathrm{x}^0 + a_\mu^5\mathrm{x}^1 - a_\mu^4\mathrm{x}^2 + a_\mu^0\mathrm{x}^3)\mathrm{dx}^\mu. \tag{4.75}$$

Since Christoffel symbols $\mathbf{\Gamma}_{\beta\gamma}^\alpha$ are defined as

$$\mathrm{x}'^\alpha = \mathrm{x}^\alpha + \mathrm{dx}^\alpha + \mathbf{\Gamma}_{\beta\gamma}^\alpha\mathrm{x}^\beta\mathrm{dx}^\gamma, \tag{4.76}$$

we thus have:

$$\mathbf{\Gamma}_{0\mu}^0 = \mathbf{\Gamma}_{1\mu}^1 = \mathbf{\Gamma}_{2\mu}^2 = \mathbf{\Gamma}_{3\mu}^3 = 2a_\mu^0, \tag{4.77}$$

$$\mathbf{\Gamma}_{0\mu}^1 = \mathbf{\Gamma}_{1\mu}^0 = 2a_\mu^1\ ;\ \ \mathbf{\Gamma}_{0\mu}^2 = \mathbf{\Gamma}_{2\mu}^0 = 2a_\mu^2\ ;\ \ \mathbf{\Gamma}_{0\mu}^3 = \mathbf{\Gamma}_{3\mu}^0 = 2a_\mu^3, \tag{4.78}$$

$$\mathbf{\Gamma}_{3\mu}^2 = -\mathbf{\Gamma}_{2\mu}^3 = 2a_\mu^4\ ;\ \ \mathbf{\Gamma}_{1\mu}^3 = -\mathbf{\Gamma}_{3\mu}^1 = 2a_\mu^5\ ;\ \ \mathbf{\Gamma}_{2\mu}^1 = -\mathbf{\Gamma}_{1\mu}^2 = 2a_\mu^6. \tag{4.79}$$

First remark, the connection of $\mathbf{\Gamma}_{\beta\gamma}^\alpha$ symbols depends only on 28 of the 32 real functions contained in the dilator M in (4.66). The four a_μ^7 are necessarily absent in the connection, because they are factors of the i generator of the chiral gauge [22][24]. Differential geometry cannot perceive these a^7_μ! Einstein thought that something was lacking in the physical theory for the integration of quantum physics into classical physics. The four parameters that are lacking in the geometric part of the connection are not lacking in the spinor part of differential geometry. Thus something was actually lacking, only not where it was expected. [2]

Second remark, the equalities (4.77) to (4.79) are identical to the equalities between $\Gamma^\alpha_{\beta\mu}$ in (4.26) and (4.27). The identity between these two connections is also another way [3] to identify inertia with gravitation, with the 64 equalities:

$$\mathbf{\Gamma}^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta\mu}. \tag{4.80}$$

There are actually only $7 = 28/4$ independent equations:

$$2a^1_\mu = \Gamma^0_{1\mu} = D_\mu \cdot [\mathcal{S}_{(1)} - 2q\mathcal{A}_{(2)}] + 2m_g\rho\delta^2_\mu, \tag{4.81}$$
$$2a^2_\mu = \Gamma^0_{2\mu} = D_\mu \cdot [\mathcal{S}_{(2)} + 2q\mathcal{A}_{(1)}] - 2m_g\rho\delta^1_\mu, \tag{4.82}$$
$$2a^3_\mu = \Gamma^0_{3\mu} = D_\mu \cdot \mathcal{S}_{(3)}, \tag{4.83}$$
$$2a^4_\mu = \Gamma^2_{3\mu} = -D_\mu \cdot [\mathcal{S}'_{(1)} - 2q\mathcal{A}'_{(2)}], \tag{4.84}$$
$$2a^5_\mu = \Gamma^3_{1\mu} = -D_\mu \cdot [\mathcal{S}'_{(2)} + 2q\mathcal{A}'_{(1)}], \tag{4.85}$$
$$2a^6_\mu = \Gamma^1_{2\mu} = -D_\mu \cdot [\mathcal{S}'_{(3)} + 2qA] - 2m_g\rho\delta^0_\mu, \tag{4.86}$$
$$2a^0_\mu = \Gamma^0_{0\mu} = D_\mu \cdot [-2\mathcal{T} + 2q\mathcal{A}'_{(3)}]. \tag{4.87}$$

We again see clearly seven a^n, $n = 0, 1, \ldots, 6$ vectors, with a^7 lacking. These vectors will be called inertial potentials. These 7 equations may be considered the gravitational field equations.

Vectors transforming like (4.76) are called contravariant. On the other hand, covariant vectors transform like ∇ :

$$\nabla = \sigma^\mu \partial_\mu = \overline{M}\sigma^\mu \widehat{M}\partial'_\mu, \tag{4.88}$$

with always the same σ^μ. These relations, demonstrated in A.4.4, do not place the ∂'_μ operators behind $\widehat{M}$ but before, because M is taken to be constant. Nevertheless for a variable M, it is (4.88) that is proved in A.4.4, because the proof only uses algebraic properties of partial derivatives. This

2. This number, $28 = 8\times 7/2$, is also the dimension of the $SO(8)$ group of the rotations in Cl_3. $36 = 64 - 28 = 8 \times 9/2$ is the number of densities that can be constructed from the electron wave [17].

3. We are accustomed to the formulation of general relativity as the equality between the Ricci tensor and the energy–momentum tensor of the other forces. But Einstein also studied a theory of the space-time manifold with torsion [106] that was very close to our approach, which is necessary if we begin with the Standard Model, and also necessary if we want that both inertia and gravitation may be defined from the unitary field that is for us the quantum wave. Since the equivalence principle identifies two connections, the tensors of torsion and curvature are also unified.

gives:

$$\nabla\widehat{\phi} = \sigma^\mu \partial_\mu \widehat{\phi} = \overline{M}\sigma^\mu \widehat{M}\partial'_\mu \widehat{\phi}$$
$$= \overline{M}\sigma^\mu [\partial'_\mu(\widehat{M}\widehat{\phi}) - (\partial'_\mu \widehat{M})\widehat{\phi}]. \tag{4.89}$$

And we have:

$$(\partial'_\mu \widehat{M})\widehat{M}^{-1} + \widehat{M}(\partial'_\mu \widehat{M}^{-1}) = \partial'_\mu(\widehat{M}\widehat{M}^{-1}) = \partial'_\mu(1) = 0,$$
$$\partial'_\mu \widehat{M} = -\widehat{M}(\partial'_\mu \widehat{M}^{-1})\widehat{M}. \tag{4.90}$$

If we define the derivative $\mathbf{D}$ as

$$\overline{\phi}\mathbf{D}\widehat{\phi} = \overline{\phi}[\nabla - \frac{1}{2}(\nabla\widehat{M}^{-1})\widehat{M}]\widehat{\phi}, \tag{4.91}$$

we necessarily have

$$\overline{\phi}'\mathbf{D}'\widehat{\phi}' = \overline{\phi}'\Big[\nabla' - \frac{1}{2}(\nabla'\widehat{M}'^{-1})\widehat{M}'\Big]\widehat{\phi}', \tag{4.92}$$
$$M' = M^{-1}, \tag{4.93}$$
$$\overline{\phi}'\mathbf{D}'\widehat{\phi}' = \overline{\phi}'\Big[\nabla' - \frac{1}{2}(\nabla'\widehat{M})\widehat{M}^{-1}\Big]\widehat{\phi}'. \tag{4.94}$$

We therefore get:

$$\overline{\phi}'(\mathbf{D}'\widehat{\phi}') = \overline{\phi}'\Big[\nabla' - \frac{1}{2}(\nabla'\widehat{M})\widehat{M}^{-1}\Big]\widehat{\phi}' = \overline{\phi}\,\overline{M}\sigma^\mu\Big[\partial'_\mu(\widehat{M}\widehat{\phi}) - \frac{1}{2}(\partial'_\mu\widehat{M})\widehat{\phi}\Big]$$
$$= \overline{\phi}\,\overline{M}\sigma^\mu\Big[(\partial'_\mu\widehat{M})\widehat{\phi} + \widehat{M}(\partial'_\mu\widehat{\phi}) - \frac{1}{2}(\partial'_\mu\widehat{M})\widehat{\phi}\Big] \tag{4.95}$$
$$= \overline{\phi}\,\overline{M}\sigma^\mu\Big[\widehat{M}\partial'_\mu\widehat{\phi} + \frac{1}{2}(\partial'_\mu\widehat{M})\widehat{\phi}\Big] = \overline{\phi}(\overline{M}\sigma^\mu\widehat{M}\partial'_\mu\widehat{\phi}) - \frac{1}{2}\overline{\phi}(\overline{M}\sigma^\mu\widehat{M}\partial'_\mu\widehat{M}^{-1})\widehat{M}\widehat{\phi}$$
$$= \overline{\phi}[\nabla - \frac{1}{2}(\nabla\widehat{M}^{-1})\widehat{M}]\widehat{\phi} = \overline{\phi}(\mathbf{D}\widehat{\phi}).$$

We may then say that $\mathbf{D}$ is **form-invariant**. In a shortened form we name $\mathbf{D}$ the invariant derivative. Using the reversion and the $M \mapsto \widehat{M}$ conjugation we have:

$$\widehat{\mathbf{D}}\phi = [\widehat{\nabla} - \frac{1}{2}(\widehat{\nabla}M^{-1})M]\phi, \tag{4.96}$$
$$\overline{\phi}\widetilde{\mathbf{D}} = (\overline{\phi}\nabla) - \frac{1}{2}\overline{\phi}\,\overline{M}(\overline{M}^{-1}\nabla), \tag{4.97}$$
$$\widetilde{\phi}\overline{\mathbf{D}} = (\widetilde{\phi}\widehat{\nabla}) - \frac{1}{2}\widetilde{\phi}\widetilde{M}(\widetilde{M}^{-1}\widehat{\nabla}). \tag{4.98}$$

The quantum wave in a non-null gravitational field follows exactly the same invariant wave equations as in a null field. The only

difference: the differential operator ∇ is replaced by the invariant $\mathbf{D}$. This uses the eight space-time vectors a^n in (4.66):

$$a^n = \sigma^\mu a^n_\mu, \tag{4.99}$$

$$\mathbf{D}\widehat{\phi} = \Big[\nabla - \frac{1}{2}(\nabla\widehat{M}^{-1})\widehat{M}\Big]\widehat{\phi} \tag{4.100}$$

$$= \Big[\nabla - \frac{1}{2}(a^0 - a^1\sigma_1 - a^2\sigma_2 - a^3\sigma_3 + a^4 i\sigma_1 + a^5 i\sigma_2 + a^6 i\sigma_3 - a^7 i)\Big]\widehat{\phi}.$$

Here we must consider all 32 functions, including the four that are not implied in the calculation of the tensors of general relativity.

Under a similitude that comes from a fixed dilator N such that:

$$\underline{\mathrm{x}} = N\mathrm{x}N^\dagger;\ \nabla = \overline{N}\underline{\nabla}\widehat{N}, \tag{4.101}$$

we must have with the covariance of ∇ and of the gauge terms:

$$\mathbf{D} = \overline{N}\underline{\mathbf{D}}\widehat{N};\ \nabla\widehat{M}^{-1} = \overline{N}\underline{\nabla}(\widehat{M}\widehat{N}^{-1})^{-1}. \tag{4.102}$$

Thus with

$$\underline{M} = MN^{-1};\ \widehat{\underline{M}} = \widehat{M}\widehat{N}^{-1};\ \widehat{M} = \widehat{\underline{M}}\widehat{N}, \tag{4.103}$$

we have:

$$(\nabla\widehat{M}^{-1})\widehat{M} = \overline{N}(\underline{\nabla}\,\widehat{\underline{M}}^{-1})\widehat{M} = \overline{N}(\underline{\nabla}\,\widehat{\underline{M}}^{-1})\,\widehat{\underline{M}}\widehat{N}, \tag{4.104}$$

$$\mathbf{D} = \overline{N}\underline{\nabla}\widehat{N} - \frac{1}{2}[\overline{N}(\underline{\nabla}\,\widehat{\underline{M}}^{-1})\widehat{\underline{M}}\widehat{N}] = \overline{N}\underline{\mathbf{D}}\widehat{N}, \tag{4.105}$$

$$\underline{\mathbf{D}} = \underline{\nabla} - \frac{1}{2}(\underline{\nabla}\,\widehat{\underline{M}}^{-1})\widehat{\underline{M}}. \tag{4.106}$$

We also recall that with any M transforming x into $\mathrm{x}' = M\mathrm{x}M^\dagger$ we have $\phi' = M\phi$, and with the X of the invariant space-time in (1.279):

$$\mathrm{x}' = M\mathrm{x}M^\dagger = M(\phi X\phi^\dagger)M^\dagger = (M\phi)X(M\phi)^\dagger = \phi' X\phi'^\dagger. \tag{4.107}$$

Thus the general element X is independent of ϕ, and thus the set of X may still be called the invariant space-time.

4.2 Invariant wave equations

In the invariant derivative there are two terms containing i: b and a^7, because this i which governs the orientation of space commutes with any element of the Cl_3 algebra (reason: ordinary space has an odd number of dimensions). Thus we have no reason to distinguish a gauge transformation acting by multiplication on the right side, from a transformation acting by multiplication on the left side. We must therefore identify these two transformations with each other and use a single gauge potential. We suppose:

$$0 = a^7 + \mathrm{b} \tag{4.108}$$

This unification makes use of the incorporation of charges into potentials advocated by Socroun [102]. We may also say: when these constants are integrated into potentials, gravitation is completely at equality with gauge forces. We will obtain wave equations in curved space or with gravitation (it is the same thing) via replacing partial derivatives by derivatives completed with gauge potentials and connection terms. Since the wave equations studied in previous chapters may be considered as approximations of the complete equations when the gravitational field is negligible, we must obtain these equations simply by suppressing $a^n,\ n=0,1,\dots,6$ which are connection terms. For the lepton wave (2.143) and (2.144) become:

$$0=\Big[\nabla-\frac{a^0}{2}+\Big(\frac{a^j}{2}-i\frac{a^{3+j}}{2}\Big)\sigma_j+i(\mathrm{b}+3\mathrm{w}^3+\mathbf{l}\mathrm{v})\Big]\widehat{L}^1, \tag{4.109}$$

$$0=\Big[\widehat{\nabla}-\frac{\widehat{a}^0}{2}+\Big(\frac{\widehat{a}^j}{2}+i\frac{\widehat{a}^{3+j}}{2}\Big)\widehat{\sigma}_j-i(2\widehat{\mathrm{b}}+\mathbf{r}\widehat{\mathrm{v}})\Big]R^1, \tag{4.110}$$

$$0=\Big[\nabla-\frac{a^0}{2}+\sigma_j\Big(\frac{a^j}{2}+i\frac{a^{3+j}}{2}\Big)+i(\mathrm{b}-3\mathrm{w}^3+m_1\mathrm{v})\Big]\overline{L}^8, \tag{4.111}$$

$$0=\Big[\widehat{\nabla}-\frac{\widehat{a}^0}{2}+\widehat{\sigma}_j\Big(\frac{\widehat{a}^j}{2}-i\frac{\widehat{a}^{3+j}}{2}\Big)-i(2p\widehat{\mathrm{b}}+m_2\widehat{\mathrm{v}})\Big]\widetilde{R}^8. \tag{4.112}$$

Hence this comes from:

$$0=\Big[\nabla-\frac{a^0}{2}-i\frac{\mathrm{b}}{2}+\Big(\frac{a^j}{2}-i\frac{a^{3+j}}{2}\Big)\sigma_j+i\Big(\frac{3}{2}\mathrm{b}+3\mathrm{w}^3+\mathbf{l}\mathrm{v}\Big)\Big]\widehat{L}^1, \tag{4.113}$$

$$0=\Big[\widehat{\nabla}-\frac{\widehat{a}^0}{2}+i\frac{\widehat{\mathrm{b}}}{2}+\Big(\frac{\widehat{a}^j}{2}+i\frac{\widehat{a}^{3+j}}{2}\Big)\widehat{\sigma}_j-i\Big(\frac{5}{2}\widehat{\mathrm{b}}+\mathbf{r}\widehat{\mathrm{v}}\Big)\Big]R^1, \tag{4.114}$$

$$0=\Big[\nabla-\frac{a^0}{2}+i\frac{\mathrm{b}}{2}+\sigma_j\Big(\frac{a^j}{2}+i\frac{a^{3+j}}{2}\Big)+i\Big(\frac{1}{2}\mathrm{b}-3\mathrm{w}^3+m_1\mathrm{v}\Big)\Big]\overline{L}^8, \tag{4.115}$$

$$0=\Big[\widehat{\nabla}-\frac{\widehat{a}^0}{2}-i\frac{\widehat{\mathrm{b}}}{2}+\widehat{\sigma}_j(\frac{\widehat{a}^j}{2}-i\frac{\widehat{a}^{3+j}}{2})-i(\frac{4p-1}{2}\widehat{\mathrm{b}}+m_2\widehat{\mathrm{v}}\Big)\Big]\widetilde{R}^8. \tag{4.116}$$

For the wave equation of quarks we obtain the following in place of (3.128) to (3.132) and for $n=2,3,4$:

$$\begin{aligned}0=\Big[\nabla-\frac{a^0}{2}+\Big(\frac{a^j}{2}-i\frac{a^{3+j}}{2}\Big)\sigma_j\\&+i\Big(-\frac{\mathrm{b}}{3}+3\mathrm{w}^3_n-3\mathrm{h}^{d3}_{L\underline{n+1}}+3\mathrm{h}^{d3}_{L\,n-1}+q_1\mathrm{v}_q\Big)\Big]\widehat{L}^n,\\0=\Big[\widehat{\nabla}-\frac{\widehat{a}^0}{2}+\Big(\frac{\widehat{a}^j}{2}+i\frac{\widehat{a}^{3+j}}{2}\Big)\widehat{\sigma}_j\\&+i\Big(\frac{2\widehat{\mathrm{b}}}{3}+3\widehat{\mathrm{h}}^{d3}_{R\underline{n+1}}-3\widehat{\mathrm{h}}^{d3}_{R\,n-1}+q_2\widehat{\mathrm{v}}_q\Big)\Big]R^n,\end{aligned} \tag{4.117}$$

$$\begin{aligned}0 &= \Big[\nabla - \frac{a^0}{2} + \sigma_j\Big(\frac{a^j}{2} + i\frac{a^{3+j}}{2}\Big)\\&\quad + i\Big(-\frac{\mathrm{b}}{3} - 3\mathrm{w}^3_n - 3\mathrm{h}^{u3}_{L\underline{n+1}} + 3\mathrm{h}^{u3}_{L\,n-1} + q_3\mathrm{v}_q\Big)\Big]\overline{L}^{3+n},\\0 &= \Big[\widehat{\nabla} - \frac{\widehat{a}^0}{2} + \widehat{\sigma}_j\Big(\frac{\widehat{a}^j}{2} - i\frac{\widehat{a}^{3+j}}{2}\Big)\\&\quad + i\Big(-\frac{4\widehat{\mathrm{b}}}{3} + 3\widehat{\mathrm{h}}^{u3}_{R\underline{n+1}} - 3\widehat{\mathrm{h}}^{u3}_{L\,n-1} + q_4\widehat{\mathrm{v}}_q\Big)\Big]\widetilde{R}^{3+n}.\end{aligned} \tag{4.118}$$

This is equivalent to:

$$\mathbf{D} := \nabla - \frac{a^0}{2} - i\frac{\mathrm{b}}{2} + \frac{1}{2}\sum_{j=1}^{3}(a^j - ia^{3+j})\sigma_j, \tag{4.119}$$

$$0 = \Big[\mathbf{D} + i\Big(\frac{\mathrm{b}}{6} + 3\mathrm{w}^3_n - 3\mathrm{h}^{d3}_{L\underline{n+1}} + 3\mathrm{h}^{d3}_{L\,n-1} + q_1\mathrm{v}_q\Big)\Big]\widehat{L}^n, \tag{4.120}$$

$$0 = \Big[\widehat{\mathbf{D}} + i\Big(\frac{\widehat{\mathrm{b}}}{6} + 3\widehat{\mathrm{h}}^{d3}_{R\underline{n+1}} - 3\widehat{\mathrm{h}}^{d3}_{R\,n-1} + q_2\widehat{\mathrm{v}}_q\Big)\Big]R^n, \tag{4.121}$$

$$0 = \Big[\widetilde{\mathbf{D}} + i\Big(-\frac{5\mathrm{b}}{6} - 3\mathrm{w}^3_n - 3\mathrm{h}^{u3}_{L\underline{n+1}} + 3\mathrm{h}^{u3}_{L\,n-1} + q_3\mathrm{v}_q\Big)\Big]\overline{L}^{3+n}, \tag{4.122}$$

$$0 = \Big[\overline{\mathbf{D}} + i\Big(-\frac{5\widehat{\mathrm{b}}}{6} + 3\widehat{\mathrm{h}}^{u3}_{R\underline{n+1}} - 3\widehat{\mathrm{h}}^{u3}_{L\,n-1} + q_4\widehat{\mathrm{v}}_q\Big)\Big]\widetilde{R}^{3+n}. \tag{4.123}$$

We can notice some similarities and differences in comparison with the lepton wave equations: terms coming from inertial potentials are the same, and gravitation works in the same manner on any material wave: **gravitation is universal**. The quark sector has more gauge terms: this comes from the fact that chromodynamics acts only on quarks, as leptons are not sensitive to strong interactions. **The quark sector seems more simple and more regular than the lepton sector**: mass terms all have the same sign whereas signs are different for right or left waves in the lepton case. The gauge terms of the $U(1)$ group are also more simplified by the invariant differential term linked to gravitation: in the quark sector, only two coefficients remain as factors of the chiral potential b: $1/6$ and $-5/6$.

4.2.1 Quantization of charges

The Standard Model employs renormalization prior to comparison between theoretical calculations and experimental values. In the case of weak interactions the success of this process requires the cancellation of "anomalies" related to chirality. This cancellation comes from the fact that the sum of all the different charges of particles in each generation is zero. Since these charges come from weak charges we will obtain this suppression of anomalies by imposing, as it is done in the Standard Model, that the sum of all coefficients of the gauge potential b is null:

$$\begin{aligned}&0=\frac{3}{2}+\frac{-5}{2}+\frac{1}{2}-\frac{4p-1}{2}+3\Big(\frac{1}{6}+\frac{1}{6}+\frac{-5}{6}+\frac{-5}{6}\Big)=-2p-4\\&p=-2.\end{aligned}\tag{4.124}$$

We remark that (4.124) should imply, in the absence of quarks, that the sum is null only if $p=0$, which means only in the case without magnetic monopoles. **Thus the magnetic monopole exists only because quarks exist**. We now have in the lepton sector:

$$0=\Big[\mathbf{D}+i\Big(\frac{3}{2}\mathrm{b}+3\mathrm{w}^3+\mathbf{l}\mathrm{v}\Big)\Big]\widehat{L}^1,\tag{4.125}$$

$$0=\Big[\mathbf{D}+i\Big(\frac{5}{2}\mathrm{b}+\mathbf{r}\mathrm{v}\Big)\Big]\widehat{R}^1,\tag{4.126}$$

$$0=\Big[\widetilde{\mathbf{D}}+i\Big(\frac{1}{2}\mathrm{b}-3\mathrm{w}^3+m_1\mathrm{v}\Big)\Big]\overline{L}^8,\tag{4.127}$$

$$0=\Big[\widetilde{\mathbf{D}}+i\Big(-\frac{9}{2}\mathrm{b}+m_2\mathrm{v}\Big)\Big]\overline{R}^8.\tag{4.128}$$

The sum of coefficients of the chiral potential is:

$$\frac{3}{2}+\frac{5}{2}+\frac{1}{2}+\frac{-9}{2}=0.\tag{4.129}$$

Then (4.124) and (4.129) imply the value $e/3$ for the sum of charges of the u and d quarks, and thus only with the choice of the coefficient $-1/3$ in the definition of P_0, which implies that the choice made there is not arbitrary, but constrained by its consequence. We may also remark in (4.120) to (4.123) that the coefficients of the potential b are the same for both left and right waves of each quark. This is also a consequence of the choice $-1/3$ in the operator P_0.

From the interaction between electron and magnetic monopole Dirac found a formula relating electric charges e and magnetic charges g to the Planck constant:

$$\frac{eg}{\hbar c}=\frac{1}{2},\tag{4.130}$$

We also have:

$$\frac{e}{2}=\frac{e^2g}{\hbar c}=\alpha g;\ g=\frac{e}{2\alpha},\tag{4.131}$$

where α is the fine structure constant. This formula, the only equality explaining the quantification of charges, has been obtained via many ways [67][86]. The smartest, from the point of view of quantum physics, was Lochak's. He used the property that, with an $1/r$ electric potential, a supplementary symmetry exists aside from rotation invariance, which transforms this invariance into $SO(4)$ invariance. The continuity of the magnetic monopole wave under the $SO(4)$ group then allowed Lochak to obtain the formula (4.130) [84] [85] [86].

The various ways [67] of obtaining the Dirac formula, including Lochak's, are all based on the same presupposition: they suppose that an electric central charge acts by a potential like B on a magnetic monopole, or that a magnetic central charge acts by a potential like A on an electric charge. The problem is: potential terms are not pure tools for calculation, they are embedded in the quantum field. And Maxwell's laws indicate clearly that an electric current creates an electric potential, not a magnetic one, and conversely. There could not be any interaction between electric and magnetic charge if an electric charge was not able to create a magnetic potential or if a magnetic charge was not able to create an A potential. Moreover only the potentials are present in the wave equations, fields are not. A magnetic monopole acts only by the potential B issued of its kind of gauge invariance. And since it does not have an electric charge, he cannot act via the potential A created by electric charges. The problem is solved because we have at hand the Weinberg–Salam angle which rotates in the complex plane (A, Z^0) of (2.205) : $\mathrm{A} + iZ^0 = e^{-i\theta_W}(B + iW)$. It results that a potential $A^0 = e/r$ created by an electric charge e satisfies:

$$B^0 = \cos(\theta_W)A^0 = \frac{\sqrt{3}e}{2r} = \frac{e'}{r};\ e' = \frac{\sqrt{3}}{2}e. \tag{4.132}$$

Thus we obtain instead the Dirac relation

$$\frac{1}{2} = \frac{e'g}{\hbar c};\ \frac{1}{\sqrt{3}} = \frac{eg}{\hbar c};\ \frac{e}{\sqrt{3}} = \frac{e^2 g}{\hbar c} = \alpha g, \tag{4.133}$$

where α is the fine structure constant. By squaring we get:

$$\frac{e^2}{3} = \alpha^2 g^2 = \frac{e^2}{3\hbar c}\hbar c = \frac{\alpha}{3}\hbar c;\ |g| = \sqrt{\frac{\hbar c}{3\alpha}};\ |e| = \sqrt{\alpha\hbar c}. \tag{4.134}$$

4.3 Double link with the Lagrangian density

To be able to obtain the same properties as in flat space-time, it is necessary to replace everywhere the partial derivatives used in the first chapters by new derivatives accounting for the covariance, or contravariance, or invariance of the objects on which the partial derivatives act. Similarly the form invariance of the wave equations needs the replacement of the ∇ operator by the invariant $\mathbf{D}$ in (4.119), with:

$$\begin{aligned}
X_0 &= a_0^0 + a_1^1 + a_2^2 + a_3^3;\ Y_0 = a_1^4 + a_2^5 + a_3^6,\\
X_1 &= a_0^1 + a_1^0 + a_3^5 - a_2^6;\ Y_1 = a_0^4 + a_2^3 - a_3^2,\\
X_2 &= a_0^2 + a_2^0 + a_1^6 - a_3^4;\ Y_2 = a_0^5 + a_3^1 - a_1^3,\\
X_2 &= a_0^3 + a_3^0 + a_2^4 - a_1^5;\ Y_3 = a_0^6 + a_1^2 - a_2^1.
\end{aligned} \tag{4.135}$$

For the left wave $L^1 = \phi(1-\sigma_3)/2$ and the right wave $R^1 = \phi(1+\sigma_3)/2$ of the electron, the form-invariant equation (1.147) becomes :

$$0 = \overline{L}^1(\mathbf{D}\widehat{L}^1)\sigma_{21} + \overline{L}^1\Big(\frac{3\mathrm{b}}{2} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v}\Big)\widehat{L}^1, \tag{4.136}$$

$$0 = \overline{R}^1(\mathbf{D}\widehat{R}^1)\sigma_{21} + \overline{R}^1\Big(\frac{5\mathrm{b}}{2} + \mathbf{r}\mathrm{v}\Big)\widehat{R}^1, \tag{4.137}$$

These equations read:

$$0 = [-i\eta^{1\dagger}(\nabla + X + iY)\eta^1 + \eta^{1\dagger}(\mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v})\eta^1](\sigma_1 - i\sigma_2), \tag{4.138}$$

$$0 = [i\widehat{\xi}^{1\dagger}(\nabla + X + iY)\widehat{\xi}^1 + \widehat{\xi}^{1\dagger}(2\mathrm{b} + \mathbf{r}\mathrm{v})\widehat{\xi}^1](\sigma_1 + i\sigma_2). \tag{4.139}$$

They are equivalent, if $X = 0$, to:

$$0 = -i\eta^{1\dagger}(\nabla\eta^1) + \eta^{1\dagger}(Y + \mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v})\eta^1, \tag{4.140}$$

$$0 = i\widehat{\xi}^{1\dagger}(\nabla\widehat{\xi}^1) + \widehat{\xi}^{1\dagger}(-Y + 2\mathrm{b} + \mathbf{r}\mathrm{v})\widehat{\xi}^1. \tag{4.141}$$

These equations obviously come from the wave equations with the form:

$$0 = -i\nabla\eta^1 + (Y + \mathrm{b} + 3\mathrm{w}^3 + \mathbf{l}\mathrm{v})\eta^1, \tag{4.142}$$

$$0 = i\nabla\widehat{\xi}^1 + (-Y + 2\mathrm{b} + \mathbf{r}\mathrm{v})\widehat{\xi}^1. \tag{4.143}$$

On the contrary, if $X \neq 0$, these equations with the usual form do not come from equations with the invariant form (4.140) and (4.141), via the Lagrange equations: if we indeed consider second sides of the invariant equations as Lagrangian densities, these densities are no longer with real value, but with complex value; this comes from $\mathbf{D}^\dagger \neq \mathbf{D}$. Henceforth with $X = 0$ and:

$$\mathbf{l}^1 := \sigma^\mu(Y_\mu + \mathrm{b}_\mu + 3\mathrm{w}^3_\mu + \mathbf{l}\mathrm{v}_\mu), \tag{4.144}$$

the left wave equation of the electron reads:

$$0 = (-i\nabla + \mathbf{l}^1)\eta^1. \tag{4.145}$$

Using the adjoint, we obtain:

$$0 = i(\nabla\eta^1)^\dagger + \eta^{1\dagger}\mathbf{l}^1. \tag{4.146}$$

If X is zero we are back in the case of the Lagrangian formalism studied in Chapter 2. Conserving the definition of the Lagrangian density, we have:

$$0 = \mathcal{L}^1_L = -i\eta^{1\dagger}(\nabla\eta^1) + \mathbf{l}^1_\mu \mathrm{D}^{1\mu}_L, \tag{4.147}$$

with a Lagrangian density which is complex, not only real. The real part, which is the Lagrangian density as before, satisfies:

$$2\Re(\mathcal{L}^1) = -i\eta^{1\dagger}\sigma^\mu(\partial_\mu\eta^1) + i(\partial_\mu\eta^{1\dagger})\sigma^\mu\eta^1 + 2\mathbf{l}^1_\mu \mathrm{D}^{1\mu}_L. \tag{4.148}$$

The double logical link between wave equations and the real Lagrangian density remains, because the X_μ terms are missing in the real Lagrangian density and the wave equation that is obtained by the use of the Lagrange equations, from the real Lagrangian density, is a complete invariant wave equation. The change from the flat space-time only comes from the imaginary part iX, the left current D_L^1 is conserved.

Fifteen similar equations exist, for the fifteen other chiral spinors of the Standard Model. From one equation to the other the $X_\mu + iY_\mu$ terms are constant (universality of gravitation), and the η^n, ∇, σ^μ and $\mathrm{D}_L^{n\mu}$ must be replaced by some ξ^n, $\widehat{\nabla}$, $\widehat{\sigma}^\mu$, $\widehat{\mathrm{D}}_R^{n\mu}$ when left waves are replaced by right waves. After this change the double logical link between wave equation and Lagrangian density is conserved, as soon as X_μ are zero. Lagrange's equations allow us, as previously, to go from Lagrangian density to wave equation in ordinary form, while the multiplication on the left side by $\overline{\eta}^n$ or $\overline{\xi}^n$ allows us to obtain the wave equations in the completely invariant form. The Lagrangian mechanism thus remains a purely algebraic process and acts with any gravitational field such that $X_\mu = 0$.

With the seven other left waves it is enough to replace the $\mathbf{l}^1$ vector by the appropriate $\mathbf{l}^n$ vector:

$$\mathbf{l}^8 = Y + \mathrm{b} - 3\mathrm{w}^3 + m_l\mathrm{v}, \tag{4.149}$$
$$\mathbf{l}^n = Y - \frac{\mathrm{b}}{3} + 3\mathrm{w}_n^3 - 3\mathrm{h}^{d3}_{L\underline{n+1}} + 3\mathrm{h}^{d3}_{L\,n-1} + m_1\mathrm{v}_q, \tag{4.150}$$
$$\mathbf{l}^{3+n} = Y - \frac{\mathrm{b}}{3} - 3\mathrm{w}_n^3 - 3\mathrm{h}^{u3}_{L\underline{n+1}} + 3\mathrm{h}^{u3}_{L\,n-1} + m_3\mathrm{v}_q, \tag{4.151}$$

for $n = 2, 3, 4$. We thus obtain:

$$0 = -i\eta^{n\dagger}(\nabla\eta^n) + i(\nabla\eta^n)^\dagger\eta^n + 2\mathbf{l}^n_\mu\mathrm{D}_L^{n\mu}, \tag{4.152}$$
$$0 = -i\partial_\mu\mathrm{D}_L^{n\mu}, \tag{4.153}$$
$$0 = \nabla\eta^n + i\mathbf{l}^n\eta^n,\ n = 1, 2, \ldots, 8. \tag{4.154}$$

Next, for the right waves we simply replace $\widehat{L}^n$ with R^n, $\overline{L}^{4+n}$ with $\widetilde{R}^{4+n}$ for $n = 1, 2, 3, 4$, and η^n with ξ^n for $n = 1, \ldots, 8$, and more we have a sign change of Y. And we use the parity conjugation $P : M \mapsto \widehat{M}$, which is the main automorphism in Cl_3. We now let, for $n = 2, 3, 4$:

$$\mathbf{r}^1 = -Y + 2\mathrm{b} + \mathbf{r}\mathrm{v};\ \mathbf{r}^8 = -Y - 4\mathrm{b} + m_r\mathrm{v}, \tag{4.155}$$
$$\mathbf{r}^n = -Y + \frac{2}{3}\mathrm{b} + 3\mathrm{h}^{d3}_{R\underline{n+1}} - 3\mathrm{h}^{d3}_{R\,n-1} + m_2\mathrm{v}_q, \tag{4.156}$$
$$\mathbf{r}^{3+n} = -Y - \frac{4}{3}\mathrm{b} + 3\mathrm{h}^{u3}_{R\underline{n+1}} - 3\mathrm{h}^{u3}_{R\,n-1} + m_4\mathrm{v}_q. \tag{4.157}$$

And we obtain:

$$0 = -i\xi^{n\dagger}(\widehat{\nabla}\xi^n) + i(\widehat{\nabla}\xi^n)^\dagger\xi^n + 2\mathbf{r}^n_\mu \mathrm{D}^{n\mu}_R, \tag{4.158}$$
$$0 = -i\partial_\mu \mathrm{D}^{n\mu}_R, \tag{4.159}$$
$$0 = \widehat{\nabla}\xi^n + i\widehat{\mathbf{r}}^n\xi^n,\ n = 1,\dots,8. \tag{4.160}$$

We see a total likeness between the left wave equations and those of right waves. They differ only by gauge terms, and the replacement of σ^μ with $\widehat{\sigma}^\mu$, and by an unexpected sign change for Y. The Lagrangian density relative to L^1 conserves exactly the form used in (2.161). Lagrange's equations thus show how the (2.165) and (2.166) equations are equivalent to the equation (2.167) of L^1. To the very old question: why does a Lagrangian mechanism exist? Does such an "extremal principle" exist, above physical laws? The answer is no, because what happens is very simple: since $\phi = R^1 + L^1$ is invertible,[4] and if the X_μ are zero, the wave equation of L^1 in the usual form (2.169) is equivalent to the invariant form of the wave equation of L^1, where the real Cliffordian part satisfies the equation $0 = \mathcal{L}^1$. The usual form (2.169) of the wave equation is equivalent to the four real numerical equations (2.164) and following, which are exactly the Lagrange equations relative to the four real variables in L^1. This is possible for each spinor wave L^n and R^n. Consequently the Lagrangian mechanism is the automatic way to go from the Lagrangian equation to each wave equation in the usual form. This works without any supplementary justification coming from an integration by parts and the cancellation of terms at the boundary of the domain of integration. It is the simple consequence of the Clifford algebra structure, a purely algebraic property which depends only on the dimension and the signature of the space-time metric, and thus on the space-time geometry. The space-time manifold inherits the Lagrangian mechanism for the electron (but only in the $X = 0$ case) from the special relativity framework, because each tangent space-time to the space-time manifold, at any point-event, has same dimension and same signature as the space-time of special relativity.

4.4 Energy–momentum and kinetic momentum

Conservation properties of energy–momentum and of kinetic momentum come from the invariance of the Lagrangian density under the additive translations group and under the multiplicative Cl_3^* group generalizing $SU(2)$.

4. The existence of the inverse is not general since the wave has value in a ring, which has zero divisors, not in a field. But the invertibility property is satisfied in any point for all calculated solutions of the improved wave equation. This property is strong, because the determinant of $\phi(\mathrm{x})$ is a modulus of complex number; its square is the sum of two squares, which is zero only if each of the two terms (the invariant Ω_1 and the invariant Ω_2) is zero. Moreover, the determinant being a continuous function, if it is nonzero at one point, it is necessarily nonzero in the neighborhood of this point.

Wave equations being form-invariant under both kinds of transformations, we have only a few things to change in comparison with previous chapters. In the case of the leptonic wave the T and V tensors remain defined by (2.218) and (2.219). The change is the replacement of the covariant derivatives. We thus have:

$$T^\mu_\lambda = \frac{m}{k\mathbf{l}}T^{1\mu}_{L\lambda} + \frac{m}{k\mathbf{r}}T^{1\mu}_{R\lambda} + \frac{m}{km_l}T^{8\mu}_{L\lambda} + \frac{pm}{km_r}T^{8\mu}_{R\lambda}, \tag{4.161}$$
$$V^\mu_\lambda = \frac{m}{k\mathbf{l}}T^{1\mu}_{L\lambda} - \frac{m}{k\mathbf{r}}T^{1\mu}_{R\lambda} + \frac{m}{km_l}T^{8\mu}_{L\lambda} - \frac{pm}{km_r}T^{8\mu}_{R\lambda}. \tag{4.162}$$

The energy–momentum T is hence always the sum of four terms, one for each spinor of the leptonic wave (and the sum of twelve terms in the case of the quarks). It is enough to calculate one term and to transpose the procedure to the others. We calculate the left term of the electron, and the invariance of the Lagrangian density under translations implies:

$$\mathcal{L}^1_L = \Re[-i\eta^{1\dagger}\sigma^\mu(\partial_\mu\eta^1 + i\mathbf{l}^1_\mu\eta^1)], \tag{4.163}$$
$$\begin{aligned} T^{1\mu}_{L\lambda} &= \Re[-i\eta^{1\dagger}\sigma^\mu(\partial_\lambda\eta^1 + i\mathbf{l}^1_\lambda\eta^1)] + \delta^\mu_\lambda\mathcal{L}^1_L \\ &= \Re[-i\eta^{1\dagger}\sigma^\mu(\partial_\lambda\eta^1 + i\mathbf{l}^1_\lambda\eta^1)]. \end{aligned} \tag{4.164}$$

We thus have:

$$-i\eta^{1\dagger}(\nabla\eta^1) = -i\eta^{1\dagger}(-i\mathbf{l}^1\eta^1) = -\eta^{1\dagger}\mathbf{l}^1_\mu\sigma^\mu\eta^1 = -\mathbf{l}^1_\mu\mathrm{D}^{1\mu}_L, \tag{4.165}$$
$$2T^{1\mu}_{L\lambda} = -i\eta^{1\dagger}\sigma^\mu\partial_\lambda\eta^1 + i(\partial_\lambda\eta^{1\dagger})\sigma^\mu\eta^1 + 2\mathbf{l}^1_\lambda\mathrm{D}^{1\mu}_L. \tag{4.166}$$

Next we use the wave equation of η^1, which gives:

$$\nabla\eta^1 = -i\mathbf{l}^1\eta^1;\ \partial_\mu\mathrm{D}^{1\mu}_L = -X_\mu\mathrm{D}^{1\mu}_L, \tag{4.167}$$
$$\begin{aligned} 2\partial_\mu T^{1\mu}_{L\lambda} &= -i(\nabla\eta^1)^\dagger\partial_\lambda\eta^1 - i\eta^{1\dagger}\partial_\lambda(\nabla\eta^1) \\ &\quad + i\partial_\lambda(\nabla\eta^1)^\dagger\eta^1 + i(\partial_\lambda\eta^{1\dagger})\nabla\eta^1 + (\partial_\mu\mathbf{l}^1_\lambda)\mathrm{D}^{1\mu}_L \end{aligned} \tag{4.168}$$

We thus obtain:

$$\partial_\mu T^{1\mu}_{L\lambda} = (\partial_\mu\mathbf{l}^1_\lambda - \partial_\lambda\mathbf{l}^1_\mu)\mathrm{D}^{1\mu}_L, \tag{4.169}$$
$$\partial_\mu T^{1\mu}_L = [(\partial_\mu\mathbf{l}^1_\lambda - \partial_\lambda\mathbf{l}^1_\mu)\mathrm{D}^{1\mu}_L]\sigma^\lambda. \tag{4.170}$$

Similarly, we obtain for the other parts of the lepton wave:

$$\partial_\mu T^{1\mu}_R = [(\partial_\mu\mathbf{r}^1_\lambda - \partial_\lambda\mathbf{r}^1_\mu)\mathrm{D}^{1\mu}_R]\sigma^\lambda, \tag{4.171}$$
$$\partial_\mu T^{8\mu}_L = [(\partial_\mu\mathbf{l}^8_\lambda - \partial_\lambda\mathbf{l}^8_\mu)\mathrm{D}^{8\mu}_L]\sigma^\lambda, \tag{4.172}$$
$$\partial_\mu T^{8\mu}_R = [(\partial_\mu\mathbf{r}^8_\lambda - \partial_\lambda\mathbf{r}^8_\mu)\mathrm{D}^{8\mu}_R]\sigma^\lambda. \tag{4.173}$$

Adding the four parts of the lepton wave, we obtain:

$$\partial_\mu T^\mu = \frac{m}{k\mathbf{l}}[(\partial_\mu \mathbf{l}^1_\lambda - \partial_\lambda \mathbf{l}^1_\mu)\mathrm{D}^{1\mu}_L]\sigma^\lambda + \frac{m}{k\mathbf{r}}[(\partial_\mu \mathbf{r}^1_\lambda - \partial_\lambda \mathbf{r}^1_\mu)\mathrm{D}^{1\mu}_R]\sigma^\lambda \quad (4.174)$$
$$+ \frac{m}{km_l}[(\partial_\mu \mathbf{l}^8_\lambda - \partial_\lambda \mathbf{l}^8_\mu)\mathrm{D}^{8\mu}_L]\sigma^\lambda + \frac{pm}{km_r}[(\partial_\mu \mathbf{r}^8_\lambda - \partial_\lambda \mathbf{r}^8_\mu)\mathrm{D}^{8\mu}_R]\sigma^\lambda$$
$$= (\partial_\mu Y_\lambda - \partial_\lambda Y_\mu)\Big(\frac{m}{k\mathbf{l}}\mathrm{D}^{1\mu}_L - \frac{m}{k\mathbf{r}}\mathrm{D}^{1\mu}_R + \frac{m}{km_l}\mathrm{D}^{8\mu}_L - \frac{m}{km_r}\mathrm{D}^{8\mu}_R\Big)$$
$$+ [\partial_\mu(\mathrm{b}_\lambda + 3\mathrm{w}^3_\lambda) - \partial_\lambda(\mathrm{b}_\mu + 3\mathrm{w}^3_\mu)]\frac{m}{k\mathbf{l}}\mathrm{D}^{1\mu}_L$$
$$+ [\partial_\mu(2\mathrm{b}_\lambda) - \partial_\lambda(2\mathrm{b}_\mu)]\frac{m}{k\mathbf{r}}\mathrm{D}^{1\mu}_R \quad (4.175)$$
$$+ [\partial_\mu(\mathrm{b}_\lambda - 3\mathrm{w}^3_\lambda) - \partial_\lambda(\mathrm{b}_\mu - 3\mathrm{w}^3_\mu)]\frac{m}{km_l}\mathrm{D}^{8\mu}_L$$
$$+ [\partial_\mu(-4\mathrm{b}_\lambda) - \partial_\lambda(-4\mathrm{b}_\mu)]\frac{m}{k\mathbf{r}}\mathrm{D}^{1\mu}_R$$
$$+ \frac{m}{k}(\partial_\mu \mathrm{v}_\lambda - \partial_\lambda \mathrm{v}_\mu)(\mathrm{D}^{1\mu}_L + \mathrm{D}^{1\mu}_R + \mathrm{D}^{8\mu}_L + \mathrm{D}^{8\mu}_R).$$

Only one term more appears, compared with what we have obtained in Chapter 2: the first term, with a curvature field:

$$C_{\mu\nu} = \partial_\mu Y_\lambda - \partial_\lambda Y_\mu. \quad (4.176)$$

This field is not linked to the probability current, but to a similar current, distinguishing the role of right and left waves:

$$\mathrm{K}_l := \frac{m}{k\mathbf{l}}\mathrm{D}^1_L - \frac{m}{k\mathbf{r}}\mathrm{D}^1_R + \frac{m}{km_l}\mathrm{D}^8_L - \frac{m}{km_r}\mathrm{D}^8_R. \quad (4.177)$$

We thus get, on the place of (2.246):

$$\partial_\mu T^\mu = \Big[qF^e_{\mu\lambda}\Big(\underline{\mathrm{J}}^\mu + \frac{mp}{km_r}\mathrm{D}^{8\mu}_R\Big) + C_{\mu\lambda}\mathrm{K}^\mu_l \quad (4.178)$$
$$+ iqF^m_{\mu\lambda}\Big(\frac{m}{k\mathbf{l}}\mathrm{D}^{1\mu}_L - \frac{m}{k\mathbf{r}}\mathrm{D}^{1\mu}_R - 2\frac{m}{km_l}\mathrm{D}^{8\mu}_L - p\frac{m}{km_r}\mathrm{D}^{8\mu}_R\Big) + \frac{m}{k}G_{\mu\lambda}\mathrm{J}^\mu_l\Big]\sigma^\lambda.$$

If the electron is lone, if weak interactions are not at play, and neither C nor G fields, it remains:

$$\partial_\mu T^\mu = qF^e_{\mu\lambda}\Big(\frac{m}{k\mathbf{l}}\mathrm{D}^{1\mu}_L + \frac{m}{k\mathbf{r}}\mathrm{D}^{1\mu}_R\Big)\sigma^\lambda, \quad (4.179)$$

This gives the Lorentz force (1.305) acting on the electric current $\mathrm{j}_e = e(\frac{m}{k\mathbf{l}}\mathrm{D}^1_R + \frac{m}{k\mathbf{r}}\mathrm{D}^1_L)$ of the electron. **We truly obtain classical electromagnetism at the limit of low gravitational field**.

4.4.1 Probability density

The T^0_0 component of the energy–momentum tensor satisfies:

$$kT^0_0 = \Re\Big[-i\Big(\frac{m}{\mathbf{l}}\eta^{1\dagger}D_0\eta^1 + \frac{m}{\mathbf{r}}\xi^{1\dagger}D_0\xi^1 + \frac{m}{m_l}\eta^{8\dagger}D_0\eta^8 + \frac{m}{m_r}\xi^{8\dagger}D_0\xi^8\Big)\Big]. \quad (4.180)$$

For a solution of the wave equation with energy E of the complete wave, such as:

$$-iD_0\xi^1 = \frac{E}{\hbar c}\xi^1(\mathrm{x});\ -iD_0\xi^8 = \frac{E}{\hbar c}\xi^8(\mathrm{x}), \tag{4.181}$$

$$-iD_0\eta^1 = \frac{E}{\hbar c}\eta^1(\mathrm{x});\ -iD_0\eta^8 = \frac{E}{\hbar c}\eta^8(\mathrm{x}). \tag{4.182}$$

We then have, like in Chapter 2:

$$\begin{aligned} T_0^0 &= \frac{E}{\hbar c}\Big(\frac{m}{k\mathbf{l}}\eta^{1\dagger}\eta^1 + \frac{m}{k\mathbf{r}}\xi^{1\dagger}\xi^1 + \frac{m}{km_l}\eta^{8\dagger}\eta^8 + \frac{m}{km_r}\xi^{8\dagger}\xi^8\Big) \\ &= \frac{E}{\hbar c}\Big(\frac{m}{k\mathbf{l}}D_L^1 + \frac{m}{k\mathbf{r}}\mathrm{D}_R^1 + \frac{m}{km_l}\mathrm{D}_L^8 + \frac{m}{km_r}\mathrm{D}_R^8\Big)^0 = \frac{E}{\hbar c}\underline{\mathrm{J}}^0, \end{aligned} \tag{4.183}$$

noting always the weighted currents $\underline{\mathbf{J}}$ with relative weights $\frac{m}{k\mathbf{l}}$, $\frac{m}{k\mathbf{r}}$, $\frac{m}{km_l}$ and $\frac{m}{km_r}$. The reason of the existence of a probability in quantum mechanics remains thus the equivalence between inertial and gravitationnal mass, which implies:

$$E = \iiint dvT_0^0\ ;\quad \iiint \frac{\underline{\mathbf{J}}^0}{\hbar c}dv = 1. \tag{4.184}$$

We will note that, if the X_μ terms are no longer negligible, which make the wave equations go out of the Lagrangian case, neither energy–momentum nor currents remain conservative. These X_μ terms may be at play with the strong gravitational field around black holes. Even in the case of a weak field, the fact that the C field acts on a difference between left and right currents may have been important for the preference of weak interactions for left waves.

4.4.2 Quantization of the kinetic momentum

The approach is exactly that of section 2.5. The invariance under the Cl_3^* group of the energy–momentum tensor V, together with the normalization of the probability current, leads to the quantization of the kinetic momentum with the value $\hbar/2$, in conformity with what we know since 1926.

We saw in Chapter 3 how we may extend this quantization to the quarks. To account for gravitation it is enough to replace d_μ with D_μ in (4.119). The energy–momentum tensors and the quantization of the kinetic momentum do not change with respect to Chapter 3.

4.5 The Pauli Principle

We will here supplement what we discussed in 1.5.8 and in [31].

We recall that the Pauli principle was formulated even before the discovery of wave equations in quantum mechanics. In the framework of Bohr's model, the principal quantum number and the integer angular momentum due to rotational invariance were not enough to characterize the electron states. Afterwards, the relativistic model of a particle revolving around the nucleus gave rise to Sommerfeld's relativistic energy levels formula (**??**). To obtain rays of light as differences between these energy levels requires the use of a "total kinetic momentum" that is not an integer (always half an odd integer). Therein lies a contradiction between the mathematical certainties of group theory, which implies integer numbers for the rotation group, and well-established experimental results concerning the emission and absorption of light. This contradiction is easy to detect in a popular encyclopedia like Wikipedia (most serious course books are unhappily not much better). We now cite and translate the French version of Wikipedia: "In 1925 Pauli proposed a principle saying that several electrons cannot simultaneously be in the same quantum state. Afterwards this principle was generalized to any fermion or particle with a half-integer spin." The same encyclopedia, in its English version, tells a much more explicit but actually alternative story: "In the case of electrons in atoms, it can be stated as follows: it is impossible for two electrons of a poly-electron atom to have the same values of the four quantum numbers: $\mathbf{n}$, the principal quantum number, l, the angular momentum quantum number, m_l, the magnetic quantum number, and m_s, the spin quantum number. For example, if two electrons reside in the same orbital, and if their $\mathbf{n}$, l, and m_l values are the same, then their m_s must be different, and thus the electrons must have opposite half-integer spin projections of $1/2$ and $-1/2$". We now explain how these versions differ.

The Pauli principle was expressed in the framework of the Bohr atom where the electron was a corpuscle following particular orbits determined by these quantum numbers. This model allowed Bohr not only to understand the possible states, but he was also able to calculate energy levels corresponding to each trajectory. For instance an atom with Z protons ionized $Z-1$ times had the following energy levels:

$$E = -\frac{Z^2}{\mathbf{n}^2} 13.6\text{eV}, \tag{4.185}$$

where $\mathbf{n}$ is the principal quantum number. This result was obtained by equating the mechanical and electrical forces that act on an electron revolving around a nucleus with an atomic number Z. The Bohr model was afterwards explained in a completely different manner by the matrix quantum mechanics of Heisenberg (his results were fully compatible with this formula), and then by the wave mechanics of de Broglie and Schrödinger. The Schrödinger equation had some difficulties concerning this lone electron around a nucleus. The formula of the energy levels was not in Z^2 but gave instead:

$$E = -\frac{Z}{\mathbf{n}^2} 13.6\text{eV}. \tag{4.186}$$

As a result of divergences between theory and experiments like the one above, the quantum wave had serious difficulties in matching Heisenberg's matrix mechanics. To describe systems of electrons the wave mechanics of Schrödinger used the Pauli principle along with a perturbation method. The wave function of a system of two electrons without interaction is antisymmetric [5]:

$$\psi = \frac{1}{2}[\psi(x_1, y_1, z_1, t)\psi(x_2, y_2, z_2, t) - \psi(x_2, y_2, z_2, t)\psi(x_1, y_1, z_1, t)], \quad (4.187)$$

where t is time, (x_1, y_1, z_1) are coordinates of the position of the first electron and (x_2, y_2, z_2) are coordinates of the position of the second electron. The result of Schrödinger's approach is: the wave propagates not in space-time but in a configuration space with $3n+1$ dimensions if n electrons are at play. The wave is then reduced to a pure tool for the calculation of probabilities.

But the orbital moment and the spin quantum number m_s in the spin-up, spin-down version of the Pauli principle (the previous English version of Wikipedia) are **purely fictitious, nonphysical**. The only quantum numbers which are **measurable** (this has a precise and definite meaning in quantum physics) are: the n number which is the degree of the radial polynomial functions, $j = |\kappa| - 1/2$ because $j(j+1)$ is a proper vector of the J^2 operator of kinetic momentum, and the magnetic quantum number which was formerly denoted as m and that we relabel as λ in Appendix C, so as to differentiate it from mass. This quantum number λ is the proper value of the J_3 kinetic momentum operator (see C.2). This proper value has not only two values but $2j+1$ values: $-j,\ -j+1, \ldots, j-1, j$. There are actually two kinds of states with the Dirac equation, linear or improved (see 1.5.7), but they are not spin up – spin down states, but rather states with positive κ (there are $\mathbf{n}(\mathbf{n}+1)$ such states) or with negative κ (there are $\mathbf{n}(\mathbf{n}-1)$ such states) (more details in 1.5.7). These states have two-to-two correspondence, but not from one sign of κ to the other: they have two-to-two correspondence by opposite values of λ. [6] The true exclusion principle can use only true quantum numbers of a given scenario. For electrons in an atom this corresponds to the orthogonalization of states: we saw in 1.5.8 how for two electrons around a nucleus this corresponds to the addition of energies and of currents, to obtain a solution near the sum or difference of solutions corresponding to each electron. It is thus the same for the kinetic momentum which adds or subtracts. We explained in 1.5.8 why the Pauli principle, for electron states in atoms, may be reformulated as such:

5. The method of perturbations starts with what happens for electrons that do not interact, thus initially considered without electric charge, and only afterwards is the value of the charge reintroduced. But absolutely no physical means exists to remove or to modify the charge of an electron even a little, hence the situation from which the calculation starts is purely theoretical, nonphysical.

6. So the sentence "several electrons cannot be simultaneously in the same quantum state" is true, not the sentence containing an integer angular momentum l which is only available with the nonrelativistic approximation of the Dirac equation by the Pauli equation: Pauli's spin-up, spin-down idea, even though very popular in course books for chemistry, is only a pretty tale.

two electron states must occupy orthogonal states for the Euclidean norm $\iiint dv\underline{\mathbf{J}}^0/\hbar c$, where $\underline{\mathbf{J}}^0/\hbar c$ is the probability density. The existence of this probability means that the normalization of the electron wave is derived in 1.5.5 from the equivalence principle between inertial and gravitational mass. Now if we consider the relation existing between the energy–momentum tensor and the current:

$$T_0^0 = -\frac{E}{\hbar c}\underline{\mathbf{J}}^0. \tag{4.188}$$

The orthogonality of the states gives, for the sum over space of $\underline{\mathbf{J}}^0/\hbar c$, the value 2. We should say that the occupation number is two electrons. They are indistinguishable: we do not know which one is on each wave, because only one wave exists, not two. In a general way, for an atom with a system of electrons of total charge ne, the total energy of these n electrons is, to a good approximation, $n\,m_0c^2$: thus in Sommerfeld's energy levels formula, the proper mass must be multiplied by n. Since the electric potential term is also multiplied by n in the case of a neutral atom, the gap between levels is multiplied by n^2. This explains the Z^2 factor in energy levels in (4.185). The energy levels of an atom with several electrons do not allow us to allocate some energy to each electron separately. Moreover, the ionization energy for one electron to be displaced is calculated by the difference between the total energy of a system of n electrons and that of a system with $n-1$ electrons.

The Dirac equation, which is the linear equation approximating our improved equation, has two kinds of solutions: the ones calculated by Darwin in 1928 and that calculated by one of the present authors [13][14], which have a Yvon-Takabayasi angle everywhere defined and small. The orthogonality of the Darwin's solutions gives the normalization of Daviau's solutions and vice-versa. [7].

4.5.1 Two versions of the exclusion principle

Despite the fact that the improved wave equation is nonlinear, everything is as if the only possible waves for each electron of a two-electron system are the sum or difference of two electronic states. Therefore the wave equation for an electron of a two-electron atom is the same equation for an identical kind of wave, except that the environment, thus gauge terms and mass terms, are changed – this is essential to be able to unify all interactions.

The spin-statistics conjecture afterwards transformed the Pauli principle into another statement: the wave of a system of fermions is the antisymmetric product of these fermion waves. This is a statement on the local

7. With a Euclidean norm we indeed have for any vector:

$$\mathbf{u}\cdot\mathbf{v} = \frac{1}{4}(||\mathbf{u}+\mathbf{v}||^2 - ||\mathbf{u}-\mathbf{v}||^2)$$

.

value at a point of space-time when the condition of orthonormalization acts on the whole wave. These two statements are not equivalent. The local statement of the principle is stricter than the global statement, because the sum over all space of a null function is certainly null. But the statement with a local condition, for the Pauli principle, changes the nature of the wave, which is no longer a function of space-time in $\mathbb{C}^4$, but the difference between two products of such functions. It is the same with the nonrelativistic approximation of the Dirac equation by the Pauli wave equation, where the space of values of the wave function becomes $\mathbb{C}^2$, not the $\mathbb{C}$ field. As this space of values is not endowed with an internal product, quantum mechanics supposes the existence of a "tensor product".[8] This transforms the electron wave (a function of the 4-dimensional space-time into $\mathbb{C}^2$), for a two-electron system, into a function of a 7-dimensional space with value in $\mathbb{C}^4$, and for the electron of an n-electron system, into a function of a $(3n+1)$-dimensional space into a $\mathbb{C}^{2^n}$. This is totally unacceptable in a theory of gravitational field which cannot change the nature of this field following the number of objects with a mass. On the contrary, with a fermionic wave with value in End(Cl_3), and with an internal product which is the composition of endomorphisms we may adopt, without any mathematical difficulty, the stricter sentence: **The wave of a fermion system is the antisymmetric product of the fermion waves.**

1. Because it is compatible with the normalization of the electron waves, as a result of the equivalence principle at the basis of general relativity.

2. An electron neither changes its wave nor the type of its wave when it enters an electron system or when it exits. Only interaction terms (gauge terms and mass terms) change with the context.

3. This is possible for the spinor wave only, since orthogonality as a consequence of the normalization is usable only because the Dirac matrices are not uniquely defined. From this non-uniqueness the sum and difference of normalized orthogonal solutions obtained from one set of γ^μ matrices are normalized as well as orthogonal solutions for another set of γ'^μ matrices [13].

4. We have seen in (1.242) that with the orthonormalization the sum or difference of solutions allows us to get for the sum of the local energy density the sum of the global energies of two electrons. A consequence of the Pauli principle is then: the energy–momentum of a system of two electrons is the sum of the momentum-energies of two electrons separately considered. We may see the “normalization” part of this reasoning as firmly justified by the

8. The definition of such a product is not at all trivial, because quantum field theory supposes here, without any mathematical proof, that the properties of the tensor product (well established only for the finite-dimensional linear spaces used in classical mechanics) can account for the tensorial densities or spinor waves of relativistic quantum mechanics, and thus are supposedly able to account for the spin of a particle system (infinite-dimensional linear spaces). Moreover with (4.187) not only the set of images is changed, but also the set of departure, which is a $(3n+1)$-dimensional space. Perhaps still worse, the starting point is the nonrelativistic tale of the spin-up/spin-down theory.

equivalence principle.

5. As first pointed out by de Broglie, the local version of the exclusion principle (the anti-symmetrizing of the wave functions) gives the symmetrizing of bosonic waves as soon as a bosonic wave is built from an even number of fermionic waves [55].

The global statement is sufficient from the point of view of electrostatics, because this is enough to get the value ne for the total charge of a system of n electrons. Moreover since the scalar product employs a summation over the whole space, this justifies the existence in quantum physics of a natural nonlocality for such a sum taken at all points.

What we discuss comes from properties of the electron wave and of the $\underline{\mathbf{J}}$ current. This may be easily transposed to the $\underline{\mathbf{J}}_q$ current, and thus to a system of protons, or to a system of neutrons, or to an atomic nucleus. On the contrary, binding forces are strong enough such that the mere summation of masses is there simply a bad approximation.

4.5.2 The equivalence principle

About the extremal principle which up to now has guided the whole of mechanics and optics, we explained in 2.3.4 how this principle is not above physical laws, and that this principle simply emerges from the Clifford algebraic structure. We now see how the equivalence principle is also a consequence of properties of the wave with spin 1/2.

In the previous discussion of the Pauli principle we saw that the mass term of the wave equation is variable, in accordance with the number of particles at play. And the energy of photons emitted or absorbed is the difference between the energy levels of the system (atom, molecule, ...) before or after emission or absorption. Denoting by m_b the mass of the system considered before and m_a the mass of the transformed system, we necessarily consider the Lagrangian density as a difference. We have two other reasons for this difference: the potential b is $-a^7$ and the differential term is also easily expressed as a difference. We recall that the Lagrangian density, in the lepton case, is the sum of four terms, and in the quark case the sum of twelve terms. For the left wave of the electron, with (4.119) and (4.125) and supposing the cancellation of the X_μ, we have:

$$\begin{aligned} 0 = k\mathcal{L}^1 &= -i\eta^{1\dagger}\sigma^\mu\Big[\partial_\mu\eta^1 + i\Big(\frac{3}{2}\mathrm{b}_\mu + 3\mathrm{w}^3_\mu + \frac{a^7_\mu}{2} + \frac{1}{2}Y_\mu + \mathbf{l}\mathrm{v}_\mu\Big)\Big]\eta^1 \\ &= -k\mathcal{L}^1_i + k\mathcal{L}^1_g \end{aligned} \tag{4.189}$$

$$\mathcal{L}^1_i = \frac{i}{2}\eta^{1\dagger}\nabla\eta^1 - \eta^{1\dagger}\Big(\frac{3}{2}\mathrm{b} + 3\mathrm{w}^3 - m_a\mathrm{v}\Big)\eta^1, \tag{4.190}$$

$$\mathcal{L}^1_g = \frac{i}{2}(\partial_\mu\eta^{1\dagger})\sigma^\mu\eta^1 + \eta^{1\dagger}\Big(\frac{1}{2}a^7_\mu + \frac{1}{2}Y_\mu\sigma^\mu + m_b\mathrm{v}\Big)\eta_1, \tag{4.191}$$

$$\mathbf{l} = m_b - m_a;\ \mathcal{L}^1_i = \mathcal{L}^1_g. \tag{4.192}$$

We may notice that we did not group together the two terms containing $\mathrm{b} = -a^7$ on only one side. The fact that this potential belongs to each one of the two parts comes from this property: the multiplication by i works in the same manner whether from the right side or from the left. Thus the potential is naturally present both in the $\mathcal{L}_i^1$ part that allows us to get the forces acting on the electron, and in the $\mathcal{L}_g^1$ part that holds the a^n giving the Christoffel symbols. To these two parts of the Lagrangian density are attached two tensors of energy–momentum, equal from their definition:

$$T^{1\mu}_{Li\lambda} = \frac{i}{2}\eta^{1\dagger}\sigma^\mu(\partial_\lambda\eta^1) - \eta^{1\dagger}\sigma^\mu\Big(\frac{3}{2}\mathrm{b}_\lambda + 3\mathrm{w}^3_\lambda - m_a\mathrm{v}_\lambda\Big)\eta^1, \tag{4.193}$$

$$T^{1\mu}_{Lg\lambda} = \frac{i}{2}(\partial_\lambda\eta^{1\dagger})\sigma^\mu\eta^1 + \eta^{1\dagger}\sigma^\mu\Big(\frac{1}{2}a^7_\lambda + \frac{1}{2}Y_\lambda + m_b\mathrm{v}_\lambda\Big)\eta^1, \tag{4.194}$$

$$0 = -T^{1\mu}_{Li\lambda} + T^{1\mu}_{Lg\lambda};\ T^{1\mu}_{Li\lambda} = T^{1\mu}_{Lg\lambda} \tag{4.195}$$

Since this may be generalized for all parts of the Lagrangian density we obtain in a very general manner an equality between the inertial tensor T_i and the gravitational tensor T_g: this is the equivalence principle.

4.5.3 Mössbauer effect

A photon may be absorbed or emitted without any recoil of the nucleus, exactly as if it was emitted or absorbed by the whole crystal containing the atom and its nucleus, in spite of the fact that the frequency of the photon corresponds to a difference between energy levels of one nucleus. The understanding of this effect induces us to admit that not only is the energy–momentum tensor a difference, but also the proper mass at play in the definition of the quantum wave:

$$m_a = m_T;\ m_b = m_{ST}, \tag{4.196}$$

where m_T is the total mass (eventually that of the whole universe if it is necessary) and m_{ST} the mass of the subtotal, which is the previous sum minus that of the system that is emitting or receiving. It is well known that a frame of reference usable in quantum mechanics must be neither too massive, if we want to sidestep gravitation, nor too light because it is then impossible to neglect phenomena of recoil due to the momentum of emitted or absorbed photons[7]. It is always possible to take as the total mass that of the reference frame in which the measurements are made. Since only a difference is useful there is no problem with the immensity of masses, even if we must include stars and galaxies.

If we study a particular electron, the mass at play is the proper mass of this single electron. If the electron belongs to a system of two electrons, the mass used in the double equality $E = mc^2 = h\nu$ is the mass of the system. It is the same for the protons or the neutrons in a nucleus, or even for a crystal. This explains why the properties of a nucleus are different according to whether the nucleus is surrounded by an electron cloud or not, particularly for the probability of radioactive decay [70].

We then see that it is possible to analyze all particles and systems of relativistic quantum mechanics with physical waves propagating in a space-time whose properties are determined by these physical waves. It remains for physics indeed to go from theory to practice.

4.6 The whole space-time manifold

4.6.1 Local and global structure of space-time

By writing $\mathrm{x} = \mathrm{x}^\mu \sigma_\mu$, quantum mechanics actually includes the set of x, which is the space-time manifold, in Cl_3. Moreover, the space-time length is given by $\det(\mathrm{x}) = \mathrm{x}\overline{\mathrm{x}} = \mathrm{x} \cdot \mathrm{x}$. This equality implies that multiplication is the single operation to be considered. Thus we must also consider the physics point-of-view about length and use [42]:

$$\mathbf{x} := \frac{\mathrm{x}}{l_a};\ \mathbf{x} \in Cl_3^*. \tag{4.197}$$

The first difference with classical geometry is that the origin of the measure of time and space is at $\mathbf{x} = 1$, not 0. Second, Cl_3 is the Lie algebra of the Cl_3^* multiplicative group. This means that the neighborhood of any point O is isomorphic to Cl_3. This set is a linear space which contains two subsets: Cl_3^*, which is the set of $\mathbf{x}$ satisfying $\det(\mathbf{x}) \neq 0$, and the light cone, which is the set of $\mathbf{x}$ satisfying $\det(\mathbf{x}) = 0$. Third, these conditions exclude themselves, therefore the light cone is included in each (local) Lie algebra, not in the (global) Lie group Cl_3^*. Fourth, the only link between each Lie algebra and the whole Lie group is the exponential function, which we calculate as follows:

$$\begin{aligned}
\mathbf{x} &= a + b\mathbf{u};\ \mathbf{u} = x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3;\ (x^1)^2 + (x^2)^2 + (x^3)^2 = 1, \\
\mathbf{x}^n &= \frac{1}{2}\Big[(a+b)^n(1+\mathbf{u}) + (a-b)^n(1-\mathbf{u})\Big]
\end{aligned} \tag{4.198}$$

$$\exp(\mathbf{x}) = \sum_{n=0}^{\infty}\frac{\mathbf{x}^n}{n!} = \frac{1}{2}\Big[e^{a+b}(1+\mathbf{u}) + e^{a-b}(1-\mathbf{u})\Big] = e^a[\cosh(b) + \sinh(b)\mathbf{u}]. \tag{4.199}$$

Moreover, we have

$$\det[\exp(\mathbf{x})] = \exp[\mathrm{tr}(\mathbf{x})] = e^{2a}. \tag{4.200}$$

Thus, with $\exp(\mathbf{x}) = A + B\mathbf{u} = A + B(x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3)$ we obtain:

$$e^{2a} = \det[\exp(\mathbf{x})] = (A + B\mathbf{u})(A - B\mathbf{u}) = A^2 - B^2 \tag{4.201}$$

which implies that the light cone ($A^2 = B^2$) is the boundary of the space-time manifold and that nothing exists outside this boundary, since $e^{2a} > 0$. From this sign we may see the purely theoretical and local character of the Schwarzschild solution in general relativity. Consequently we obtain:

$$\begin{aligned} e^a &= \sqrt{A^2 - B^2};\ \cosh(b) + \sinh(b)\mathbf{u} = \frac{A + B\mathbf{u}}{\sqrt{A^2 - B^2}}. \\ a &= \ln(\sqrt{A^2 - B^2}) = \frac{1}{2}[\ln(A + B) + \ln(A - B)], \qquad (4.202) \\ b &= \sinh^{-1}\Big[\frac{B}{\sqrt{A^2 - B^2}}\Big] = \frac{1}{2}[\ln(A + B) - \ln(A - B)], \\ a + b &= \ln(A + B);\ A + B = e^{a+b}. \qquad (4.203) \end{aligned}$$

4.6.2 The EPR paradox

Two photons are emitted at point-event O. We suppose, simplifying the calculation, that they are emitted in two orthogonal directions, σ_1 and σ_2, of the tangent space-time at O. They are absorbed at the same time $y > 0$, also to simplify the calculation. The photon emitted in the direction σ_1 is absorbed at the point-event:

$$\begin{aligned} &\mathbf{x}_1 = a_1 + b_1\mathbf{u}_1 = (a + y) + (bx^1 + y)\sigma_1 + b(x^2\sigma_2 + x^3\sigma_3), \\ &a_1 = a + y;\ \mathbf{u}_1 = x_1^1\sigma_1 + x_1^2\sigma_2 + x_1^3\sigma_3;\ (x_1^1)^2 + (x_1^2)^2 + (x_1^3)^2 = 1, \\ &(x^1 + y/b)^2 + (x^2)^2 + (x^3)^2 = 1 + 2x^1 y/b + (y/b)^2, \qquad (4.204) \\ &b_1 = b\sqrt{1 + 2x^1 y/b + (y/b)^2};\ \mathbf{u}_1 = \frac{(x^1 + y/b)\sigma_1 + x^2\sigma_2 + x^3\sigma_3}{\sqrt{1 + 2x^1 y/b + (y/b)^2}}. \end{aligned}$$

The photon emitted in the direction σ_2 is absorbed at the point-event:

$$\mathbf{x}_2 = a_2 + b_2\mathbf{u}_2 = (a + y) + bx^1\sigma_1 + (bx^2 + y)\sigma_2 + bx^3\sigma_3. \qquad (4.205)$$

And we also have:

$$\begin{aligned} &a_2 = a + y;\ \mathbf{u}_2 = x_2^1\sigma_1 + x_2^2\sigma_2 + x_2^3\sigma_3;\ (x_2^1)^2 + (x_2^2)^2 + (x_2^3)^2 = 1, \\ &(x^1)^2 + (x^2 + y/b)^2 + (x^3)^2 = 1 + 2x^2 y/b + (y/b)^2, \qquad (4.206) \\ &b_2 = b\sqrt{1 + 2x^2 y/b + (y/b)^2};\ \mathbf{u}_2 = \frac{x^1\sigma_1 + (x^2 + y/b)\sigma_2 + x^3\sigma_3}{\sqrt{1 + 2x^2 y/b + (y/b)^2}}. \end{aligned}$$

On the space-time manifold, the point event O is at $\mathbf{X} = O/l_a = A + B\mathbf{u} = \exp(\mathrm{x})$ while the photon emitted in the direction σ_1 is absorbed at the point-event $\mathbf{X}_1 = M/l_a = \exp(\mathrm{x}_1)$. The photon emitted in the direction σ_2 is absorbed at the point-event $\mathbf{X}_2 = P/l_a = \exp(\mathrm{x}_2)$. The position of the point event P, seen from 1, is:

$$\mathbf{x}_2^0 = [\exp(\mathrm{x})]^{-1/2} \exp(\mathrm{x}_2)[\exp(\mathrm{x})]^{-1/2}. \qquad (4.207)$$

The position of the point event P, seen from M, is:

$$\mathbf{x}_2^1 = [\exp(\mathrm{x}_1)]^{1/2}[\exp(\mathrm{x})]^{-1/2}\exp(\mathrm{x}_2)[\exp(\mathrm{x})]^{-1/2}[\exp(\mathrm{x}_1)]^{1/2}. \tag{4.208}$$

The position of the point event M, seen from 1, is:

$$\mathbf{x}_1^0 = [\exp(\mathrm{x})]^{-1/2}\exp(\mathrm{x}_1)[\exp(\mathrm{x})]^{-1/2}. \tag{4.209}$$

The position of the point event M, seen from P, is:

$$\mathbf{x}_1^2 = [\exp(\mathrm{x}_2)]^{1/2}[\exp(\mathrm{x})]^{-1/2}\exp(\mathrm{x}_1)[\exp(\mathrm{x})]^{-1/2}\exp[(\mathrm{x}_2)]^{1/2}. \tag{4.210}$$

And we have, since the determinant of a product is the product of the determinants:

$$\begin{aligned}\det(\mathbf{x}_2^1) &= e^{a+y}e^{-a}e^{2(a+y)}e^{-a}e^{a+y} = e^{2(a+y+y)},\\ \det(\mathbf{x}_1^2) &= e^{a+y}e^{-a}e^{2(a+y)}e^{-a}e^{a+y} = e^{2(a+y+y)}.\end{aligned} \tag{4.211}$$

Therefore at each point-event, when a photon is absorbed at the local time $a + y$, each observer sees the absorption of his photon as preceding, with the same length of time y, the arrival of the photon for the other observer: the absorption of the other photon is in the future of each observer, not at the moment of arrival. This strange result seems very similar to the fact that each observer sees any length shorter for a moving object: an observer in the moving object also sees the other observer as moving, thus with shorter length. The paradox is that a measurement made on either of the particles apparently collapses the state of the entire entangled system and does so *instantaneously*, before any information about the measurement result could have been communicated to the other particle [4]. Our previous calculation shows the key to this paradox: the instantaneous character of the measurement is simply false, the "collapse" only results from the supposition that this situation may be described by a tensor product of Hilbert spaces whose elements are not defined in the mathematical sense of the word. Look out! We don't deny quantum entanglement. We say that the paradox is only in the interpretation of this situation by a nonrelativistic Hermitian theory, whereas physics must account for the fact that each "fixed" observer is journeying in time on the space-time manifold, even if he does not travel in space.

The understanding of the true geometry of space-time simply requires the use of the space-time manifold itself, not merely the use of the flat tangent space-time at the particular point-event O.

Einstein, Podolsky and Rosen said [66]: "From this follows that either (1) the quantum-mechanical description of reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality. For if both of them had simultaneous reality — and thus definite values — these values would enter into the complete description, according to the condition of completeness."

Experiments with the polarization of two photons simultaneously emitted (very fine indeed and meriting the Nobel prize for Aspect) can neither prove (1) nor (2) because the absorption of these photons cannot be simultaneous at the points where each absorption is effective. The quantum wave used here, with value in $\mathrm{End}(Cl_3)$, and not just in $\mathbb{C}$, is enough to prove that (1) was true in 1935, independently of what we now think about (2). More generally no contradiction can exist between general relativity and quantum mechanics. **Any apparent contradiction results from bad approximations of relativistic laws**.

4.6.3 The arrow of time, the expansion of the universe

Any point of the space-time manifold is at a position:

$$X = l_a \exp(a + b\mathbf{u}) = l_a(A + B\mathbf{u});\ A = e^a \cosh(b);\ B = e^a \sinh(b). \quad (4.212)$$

Thus the time position $l_a e^a \cosh(b)$ is the product of positive real numbers: time is an oriented quantity, **the arrow of time has a geometric root**. The time variable goes from 0 to $+\infty$.

Now we consider a photon received at this position X, coming from a distant galaxy, for instance along the σ_1 direction. It was emitted at the position:

$$l_a \exp[a - y + (bx^1 - y)\sigma_1 + b(x^2\sigma_2 + x^3\sigma_3)] = l_a \exp(a_1 + b_1\mathbf{u}_1), \quad (4.213)$$

with [9]

$$\begin{aligned} &a_1 = a - y;\ \mathbf{u}_1 = x_1^1\sigma_1 + x_1^2\sigma_2 + x_1^3\sigma_3;\ (x_1^1)^2 + (x_1^2)^2 + (x_1^3)^2 = 1, \\ &(x^1 - y/b)^2 + (x^2)^2 + (x^3)^2 = 1 - 2x^1y/b + (y/b)^2, \qquad (4.214) \\ &b_1 = b\sqrt{1 - 2x^1y/b + (y/b)^2};\ \mathbf{u}_1 = \frac{(x^1 - y/b)\sigma_1 + x^2\sigma_2 + x^3\sigma_3}{\sqrt{1 - 2x^1y/b + (y/b)^2}}. \end{aligned}$$

The photon was emitted at

$$x_e = l_a\, e^{a_1}[\cosh(b_1) + \sinh(b_1)\mathbf{u}_1]. \quad (4.215)$$

At this point-event the local time was $t_e = l_a e^{a_1} \cosh(b_1) \approx l_a e^{a_1+b_1}/2$. The same photon is absorbed at the point-event X, then at the local time $t_a = l_a e^a \cosh(b) \approx l_a e^{a+b}/2$. The only constant object of this geometry is the Lie algebra: each local tangent space, in each point of the manifold, is isomorphic to the Lie algebra of the group. We will then suppose that

$$d(a_1 + b_1) = d(a + b);\ \frac{dt_e}{t_e} = \frac{dt_a}{t_a} \quad (4.216)$$

9. Since we now look at past, $a_1 < a$.

And we have
$$\frac{\nu_a}{\nu_e} = \frac{dt_e}{dt_a}. \tag{4.217}$$
In the first approximation, $b_1 \approx b$, we obtain:
$$\begin{aligned} \frac{1}{1+z} = \frac{\nu_a}{\nu_e} = \frac{dt_e}{dt_a} &= \frac{d[l_a e^{a_1}\cosh(b_1)]}{d[l_a e^a \cosh(b)]} \\ &\approx \frac{l_a\, da\, e^{a-y}\cosh(b)}{l_a\, da\, e^a \cosh(b)} = \frac{1}{e^y} \approx \frac{1}{1+y}. \end{aligned} \tag{4.218}$$
This means that the redshift due to the expansion of the universe, previously interpreted as a Doppler effect, is a direct effect of the geometry of space-time, and the z parameter, defined as $(\nu_e - \nu_a)/\nu_a$, is almost equal to y. But this is true only as a crude approximation, or as a false velocity. When y is small this redshift seems proportional to y. The Hubble parameter (73.3 ± 1.4km/s/Mpc) gives for the distance 1 Mpc the value $z = 0.0002443$, thus giving $R = l_a\, e^{a+b}/2 \approx 6.3 \times 10^{25}$m.

Using the geometric condition (4.216), which results from the Lie algebra as the only fixed framework, independent from the space-time position on the manifold, we may calculate more precisely the ratio dt_e/dt_a in the case where y is small. We have:
$$\frac{d[l_a e^{a_1}\cosh(b_1)]}{d[l_a e^a \cosh(b)]} = \frac{d[e^{a-y}\cosh(b_1)]}{d[e^a \cosh(b)]} = \frac{e^{-y}\cosh(b_1)}{\cosh(b)} = \frac{1}{f(y)} \tag{4.219}$$
$$f(y) := e^y \frac{\cosh(b)}{\cosh(b_1)} \approx f(0) + y f'(0) + y^2 \frac{f''(0)}{2} + ... \tag{4.220}$$
We use:
$$\begin{aligned} b_1 := b g(y) = \sqrt{b^2 - 2x^1 b y + y^2};\ g(y) = \sqrt{1 - 2\frac{x^1}{b} y + (\frac{y}{b})^2}, \\ g(y) \approx 1 - \frac{x^1}{b} y + \frac{1-(x^1)^2}{2b^2} y^2 + \frac{x^1[1-(x^1)^2]}{2b^3} y^3 + \dots . \end{aligned} \tag{4.221}$$
And we obtain:
$$f(y) \approx e^y \frac{e^b}{e^{b_1}} = e^{a(y)} \tag{4.222}$$
$$\begin{aligned} a(y) = y + b - b_1 &\approx (1+x^1)y - \frac{1-(x^1)^2}{2b} y^2 - \frac{x^1[1-(x^1)^2]}{2b^2} y^3 + \dots, \\ f'(y) \approx a'(y) e^{a(y)} &= (1+x^1)[1 - \frac{1-x^1}{b} y - \frac{3x^1(1-x^1)}{2b^2} y^2 + \dots] e^{a(y)}. \end{aligned} \tag{4.223}$$

From the values of the Hubble parameter and of l_a we obtain $a + b \approx 142$. We only know that $a > b > 0$. The ratio a/b is unknown. If our position in the manifold is anywhere, for instance is $(a+b)/a \approx a/b$, we could have $a \approx 88$ and $b \approx 54$. This should give a ratio B/A very close to 1. We now look at the acceleration or deceleration of the expansion.

4.6.4 Beginning of the acceleration

Defining h such that $h(y) := f(y)/y$ the redshift seems accelerated if and only if h is increasing, hence if $h'(y) > 0$. We obtain:

$$\begin{aligned} y^2 h'(y) &= y f'(y) - f(y) \approx [y a'(y) - 1] e^{a(y)} \qquad (4.224) \\ &= [-1 + (1 + x^1) y - \frac{1 - (x^1)^2}{b} y^2 - \frac{3x^1[1 - (x^1)^2]}{2b^2} y^3 + \ldots] e^{a(y)}. \end{aligned}$$

For instance if $b = 40$ and $x^1 = 0.6$ we have:

$$y^2 h'(y) \approx [-1 + 1.6y - 0.016y^2 - 0.00036y^3 + \ldots] e^{a(y)} \qquad (4.225)$$

Thus, in this case, $h'(y) > 0$ if and only if

$$y > y_0, \ y_0 \approx 0.63. \qquad (4.226)$$

Moreover, the sign of the coefficient of y^3 indicates a sign change for large y, but the method of calculation used here does not give the value of this new change of sign.

Hence the acceleration of the expansion seems to begin near y_0, with possible differences depending on the direction of observation with regard to the whole space-time manifold. And the expansion seems to decelerate for very large z. Thus there is no need for either dark matter [10] or repulsive gravity to explain all modern observations of cosmological redshifts. What we obtain here is completely different from the cosmology developed since Einstein's works on relativistic gravitation. We think that it is much more satisfactory, because we do not need to suppose a homogeneous distribution of matter, never observed at small or very large scales; neither is there need for a gigantic amount of unknown matter, nor for an *ad hoc* cosmological constant, nor for an adjustment in the parameters to explain the recent acceleration of the expansion. Furthermore we have obtained two results that Einstein should have very much liked: first, a space-time which, as a whole manifold, is invariant (for a given cosmological time t, space is not an hypersphere S^3 with growing radius, it is a $\mathbb{R}^3$ unlimited and globally invariant. Only the elements of the whole manifold are variable. Second, the geometry of the cosmos, globally, is independent of matter, integrating to the geometry both inertia and gravitation.

10. The movement of stars in galaxies and the movement of galaxies in galaxy clusters is another question. Indeed the absence of necessity for the cosmology of dark matter does not prove its non-existence. The simple name "black hole" is enough to prove that some objects may exist and be unable to directly send light.

Chapter 5

Why?

Thousands of years ago, physics began with the questions of our ancestors: why the regular return of the Sun, why the phases of the Moon? Why the wind, the rain, why the rainbow after a thunderstorm? When physics started to progress more and more quickly, understanding the motion of the planets, linking together all these "whys" into a theory of gravitation, of electricity and of light, gave rise to many other whys.

Take the instance of light. Physicists began by understanding some of its **properties**, like the fact that it originated in the Sun before coming to the retina of our eyes, and not the reverse, as was long believed. Going a little further they understood some **laws** governing these properties, for instance the law of refraction when light passes from one medium to another with a different refractive index. These laws are described with mathematical tools, such as sines of angles of refraction. Next these laws depend on **principles** which are, in a sense, laws governing laws. Concerning light, Pierre Fermat understood that the law of refraction comes from the following physical principle: light automatically chooses the path of shortest duration. We previously did not only study **properties** of quantum waves (they are functions of space-time with values in Cl_3^*), we have studied **laws**: partial differential equations for the wave, also the orthonormalization of the electron wave and the existence of a probability density. We also obtained the laws of motion for a charged fluid. We have even explained how these laws come from **principles**: the wave equations arise through the Lagrangian mechanism from an extremal principle resulting from a Lagrangian density. The orthonormalization of the wave comes from the principle of equivalence between inertial mass and gravitational mass. What is newest here: to end up with a causal loop, by the deduction of these principles from the properties of matter waves themselves. We have completely dissected how the extremal principle is, for quantum waves, a consequence of properties of the quantum wave as a function with value in a particular Clifford algebra. These properties are linked to the structure of space-time, the fact that time is 1-dimensional and space is 3-dimensional. We have also explained how the equivalence principle comes from the properties of all densities of energy–momentum.

What we have continued here is proper to the building up of science: to search for laws from properties of physical objects, and not beyond these objects. The causal loop that we have just described is hence a successful realization of the scientific process for this field of science that studies matter, called physics. And as a loop closes on itself this rounds out the process, even if a loop may indeed be extended, doubled or integrated into other similar loops.

The double equality $E = mc^2 = h\nu$ is an essential component to these parts of our causal loop which have been progressively improved over time. The equality $E = mc^2$ comes from the electrodynamics of matter in motion, obtained by Albert Einstein in 1905. Straight after sending his article for publication he accounted for this: if all matter has an electromagnetic origin, then $E = mc^2$. This equality is extremely well established by experiment, but as a consequence the if-then nature of the statement is somewhat forgotten. Physicists no longer asked: why does all matter have an electromagnetic origin? Here we have carried our knowledge on this point a little further: all fundamental objects of physics are fermions obeying the same laws; thus saying that all matter has the same origin is equivalent to this: any mass-energy in the universe comes from fermions. Hence if any boson seems to have a proper mass, it is composed of fermions that possess this mass.

After his discovery of the electrodynamic laws of matter in motion, Einstein reconsidered gravitation, starting from the identity between inertial mass and gravitational mass. This identity implies that the gravitational field is an acceleration field, not a force field, unlike electromagnetic forces. He thus understood that gravitation was a completely geometric phenomenon, linked to the structure of space-time itself, its curvature. But then why was gravitation this way? Why the equivalence between inertial mass and gravitational mass? We have also advanced a little further here by showing that Lagrangian densities of fermions may naturally be interpreted as a null difference between gravitational terms and inertial terms. We have even further advanced the next question, "Why is this the case?" The different terms of the wave equations are the only possible ones, able to exist in a manner compatible with the form invariance of the wave equations. Furthermore these wave equations are form-invariant due to the properties of the structure of waves themselves. This causal loop goes through the Lagrangian mechanism that we have dissected and which contains no metaphysical principle. Everything arises from the algebraic structure automatically associated to the geometric structure of space-time. This structure is itself linked to the quantum wave, which has value on the Cl_3^* manifold that includes this space-time.

The second equality of the pair, $E = h\nu$, was first obtained by Max

Planck in his studies of the laws of the radiation issuing from a material heated at a high temperature. The equality contains a constant which is rightfully named after its discoverer. This was extended twice, first by Albert Einstein who introduced the wave-particle dualism for light as early as 1905, then by Louis de Broglie who a century ago extended this dualism to all matter[53]. Between then and now, multiple discoveries of this quantum world have taken place. They are nowadays described by the Standard Model of quantum physics.

5.1 Einstein was right

Despite his discoveries, both of wave-particle dualism and of gravitation as geometry of space-time, Einstein ended up being isolated from the scientific community: A quantum physics was developed in a very different way from the physics of gravitation. Einstein continued to search for a unifying synthesis aiming to encompass electromagnetism and quantum physics along with the physics of gravitation. He was seeking after what was so characteristic of his theory of gravitation: a completely relativistic physics, with a field following a partial differential equation, deterministic, and able to yield the laws of motion of field sources.

This is exactly what comprises the set of partial differential equations that we have obtained for the fermionic waves: they are completely deterministic and they allow us to derive the laws of motion of these sources of gauge fields, which fermions actually are. Einstein was thus right to attempt such a synthesis, as our previous chapters show it to be workable.

Why has Einstein not been understood? The first reason was the novelty of his understanding of the nature of space and time, particularly his rejection of an absolute time. Schrödinger, who himself perfectly understood the relative time of Einstein's gravitation, first found a nonrelativistic wave equation for de Broglie's wave. This wave equation, plus Pauli's exclusion principle, resulted in a wave which does not have direct physical reality. This wave does not propagate in space-time but in an absolute time and in configuration space, whose geometrical properties Einstein himself was the first to use.

From 1917 until his death Einstein made many attempts to reconcile gravitation, electromagnetism and quantum physics. He tried for instance a manifold with torsion, in a manner very close to our calculations. But his starting point was not the chiral right and left waves issuing from the discovery, just after his death, of maximal parity violation in weak interactions. Moreover, nonlocal properties of quantum waves (he first conceived of their existence) were not yet understood. Einstein was indeed not truly happy with his theory of gravitation. He had serious doubts about the longevity of his theory, notably because the left side of his equation, which is purely geometric, is much stronger than the right side, which was not uniquely defined, which may or may not include a cosmological constant. Einstein was not fortunate with his view of the whole of space-time as necessarily invariant and steady-state, while astronomers discovered the immensity of the cosmos that appeared to be expanding. There also it was Einstein's view which was right, since the space-time manifold, invariant in its totality, is perfectly compatible with the redshift of distant galaxies, and moreover with a recent acceleration of this redshift, an acceleration measured today by astronomers, and which can be explained as a purely global geometric effect (see 4.7.3).

5.1.1 "There is no alternative"

Certainly there is no way to escape the double equality $E = h\nu = mc^2$, and it should be stupid to claim the opposite. Certainly, it is impossible to avoid Heisenberg's inequalities since the kinetic momentum of all fermions is quantized. We agree all the more since we know why. But for many other things an alternative exists, and the proof is precisely our work: we have worked out laws of nonrelativistic quantum mechanics and obtained better results. A common opinion still believes the Dirac equation as "a kind of Schrödinger equation." This is false! This error persists only because of the well-hidden "sleight of hand" changing the Dirac equation into another nonrelativistic equation before the presentation of the Hamiltonian density. The electromagnetic interaction is part of a gauge interaction described by a noncommutative $U(1) \times SU(2)$ gauge group: it is therefore impossible to dissociate this interaction from other electroweak interactions.

Quantum field theory was developed from a wave with only one phase. But the electron always has two phases. Certainly the second phase, which appears in magnetic phenomena and in electroweak interactions, is very difficult to see in many situations: Only then does quantum electrodynamics work perfectly, even with its most surprising predictions.

In physics the universe is what it is. We have changed the title of this work from "Developing a Theory of Everything" in the first edition to "Developing *the* Theory of Everything"; it is another way of saying that there is no alternative. Time must be ordered, thus time is necessarily unidimensional. Space is 3-dimensional, thus the algebra of space is Cl_3. The rotational invariance of the laws of mechanics (there is no privileged direction in space) has been replaced in quantum mechanics by the invariance under the $SU(2)$ Lie group. This leads one to consider (since nearly a century ago!) space-time as included in the part of Cl_3 containing $SU(2)$, which can only be the whole multiplicative group Cl_3^*: the space-time of general relativity is a four-dimensional manifold, and Cl_3^* is just big enough (Whitney's theorem), with its eight dimensions, to include any four-dimensional manifold. The pseudometric of space-time comes from the determinant, and thus the signature of space-time is $+, -, -, -$. Cl_3^* is a Lie group, each Lie group is associated to a unique Lie algebra, and the Lie algebra of Cl_3^* is Cl_3: there is no alternative. A Lie group is a manifold and the tangent space at any point of the Lie group is isomorphic to the tangent space at the neutral element of the group, which is the Lie algebra: no alternative!

5.1.2 After this work

When we state why the Weinberg–Salam angle θ_W exactly satisfies the equality $\sin(\theta_W) = 1/2$, or why the charge of the d quark is exactly a third of the charge of the electron, the precision is certainly above eleven significant digits since it is exact. The advantages of a correct understanding concern not only the precision of predictions: Understanding why two colored quarks exist in each generation, why leptons are insensitive to strong interactions, why a Lagrangian mechanism exists, how the electromagnetic field is directly linked to the energy–momentum of the quantum wave – these are definitive advancements. The same predictive power must be expected for any proposed alternative. If such an attempt obtains, for instance, better predictive power from a Lagrangian with both an independent fermion part and a boson part, the existence of the Lagrangian itself will need to be accounted for. Indeed we gave an explanation for the existence of Lagrange equations in the fermion wave case: Thus any attempt to build a ToE will be asked to do the same. We arrived at a simple origin of light polarization: Any further ToE attempt will be asked for its ability to derive the link between the electromagnetic field and energy–momentum of the fermion wave. We also understood the geometric reason for the time arrow, and the redshift of light coming from very distant stars, including the recent acceleration of this redshift, all from the very structure of the space-time manifold: any further ToE attempt will be asked for such a simple explanation of this "expansion".

The most important and novel understanding brought by the present work is the quantization of the kinetic momenta of the electron, neutrino, proton and neutron with the same $\hbar/2$ value. From this quantization of the kinetic momentum come both Heisenberg's inequalities and the quantization of the electric charge. Any attempt to build a ToE will also be expected to obtain this quantization and with the true value, fully established experimentally.

5.2 de Broglie was right

Einstein and de Broglie were right, because the quantum wave is fundamentally relativistic. With the electron wave, whether in the case of low or of high velocity, the D_R and D_L currents, formed respectively by the right and left parts of the quantum wave, are on the light cone. These currents indeed have a sum which is the probability current, linked to the invariance of the electric gauge. This J current is the only one visible in the version of quantum mechanics at the basis of QFT. But the D_R and D_L currents also have a difference which is the second K current. This current is also present in the wave equation. It is also linked to the chiral gauge, and thus linked to magnetism and to weak interactions. The dependence of tensor densities on the wave chirality concerns not only currents; it is also extended to densities of energy–momentum and of kinetic momentum, and thus to the electromagnetic field. By accounting for this dependence, we obtained in 2.5 and 3.7 the quantization of kinetic momentum. Since the quantum wave is fundamentally relativistic, the replacement of the Dirac equation by the Pauli equation is untenable; thus the integration of the electron into Hamiltonian physics does not work out well. This is why difficulties arise in all QFT calculations, such as the infinite quantities that must be gotten around, renormalization and anomalies that must be tamed. And all this turns out to be impossible for gravitation, and justly, because gravitation is completely relativistic.

In his time Einstein could not elaborate a better theory, as he could not know discoveries made after his death, discoveries that later allowed the building of the Standard Model. The most important discovery of the second half of the last century (according to Lochak) was the violation of parity in weak interactions. The different roles of right and left waves is important as well for the Standard Model which carefully accounts for this difference, as for general relativity. This is due to the orientation of space, placed again at the center of physics. This orientation of “space” is a convenient shorthand; it is actually the orientation of space-time and the arrow of time which are conserved, and therefore the orientation of space.

De Broglie was very much aware of the defects of quantum theory that stems from nonrelativistic wave equations. He thoroughly studied relativistic Dirac theory twice, and published two books [54, 57]. He also used the Dirac wave as the starting point for his theory of light [55, 56]. These works were not understood; they were too far ahead of his time. And de Broglie could not know the existence of quarks and their chromodynamics. The present work was mainly developed by two persons who met in the seminar organized by de Broglie himself in the “Fondation Louis de Broglie” created in Paris to continue his scientific work. The director of this private foundation was G. Lochak who discovered the leptonic magnetic monopole [84, 85]. His monopole wave equation was the starting point of our work.

De Broglie not only bequeathed us his very deep knowledge of the various domains of classical and quantum physics, but also advised us to exercise our freedom in critiquing the fashions of a system where everyone takes too many unsound habits for granted.

5.3 Bohr was also (partly) right

During the early development of quantum mechanics, the universality of Heisenberg's inequalities and the implied limits of our knowledge were not at all evident. Einstein, who had encountered nothing similar to his theory of gravitation, entered into a great debate with Bohr, but Bohr's arguments prevailed, and justly because it was he who made use of the universality of Einstein's relativistic physics. The generalization of relativistic invariance to the Cl_3^* group again takes up this universality and allows us to obtain, as a consequence of the properties of fermion waves, the quantization of the kinetic momentum with a value of $\hbar/2$ (see 2.5 and 3.7).

In his second book on the Dirac theory (see [57] 2.6), from the quantization of the kinetic momentum, de Broglie deduced the precise form of the uncertainty relations for two quantities A and B canonically conjugated (like x and p_x): $\sigma_A \cdot \sigma_B \geqslant \hbar/2$. Before this book, de Broglie had studied Heisenberg's inequalities in an earlier work written in 1950-1951 but edited only thirty years later [58] thanks to Lochak. The derivation of the quantization of kinetic momentum allows us to obtain the fourth uncertainty relation in the form proved by de Broglie: $\sigma_t \cdot \sigma_E \geqslant \hbar/2$, where σ_t is the uncertainty in the temporal coordinate of an event and σ_E is the uncertainty of the energy at work in this event.

5.4 Intrinsic or statistically random?

Einstein was the first to understand Brownian motion as the random movement of a particle constantly colliding with molecules, and obviously had nothing against probabilities. What he questioned was the intrinsic randomness attributed to the quantum wave.

When physicists can afford to suppress the "small components" of the Dirac wave, just because electron velocity is low and because two of the complex components then have a small modulus in comparison with the other two, they not only completely destroy the relativistic invariance of the wave equation, but also return to the Hamiltonian pattern inhabited by the Schrödinger and Pauli equations. In that case time plays a different role in comparison with space, and the equation takes the Hamiltonian form of the Schrödinger equation $i\hbar\partial_t\psi = H(\psi)$. The wave equation obtained by this suppression is seldom presented as merely a Pauli equation. In the end, everyone believes that the Dirac equation is "a kind of Schrödinger equation." This reduces the Dirac wave to the general probabilistic schema of nonrelativistic quantum theory: the only things we may calculate are probabilities. And since there is no physical reality beyond what is measurable, the research of other ideas – the understanding of what actually happens – are considered useless, and even harmful.

Have we in the present work moved outside this probabilistic schema?

At first it may seem not, since the quantum wave with spin $1/2$ is always associated with a probability: by dividing the local energy density by the total energy we naturally arrive at a density whose summation over the whole space takes the value one. It is thus a probability density. In the case of several indistinguishable fermions we also obtain a measure which gives the number of these fermions. But certainly the answer is yes, we have gone outside the purely probabilistic schema because the wave does not give *only* probabilities. In the wave we can find the origin of all the so-called "quantum numbers" such as the baryon number, lepton number, weak hypercharge and so on. We are now able to understand the value of each elementary charge. We are also able to derive the Lorentz law of motion for the density of electric charge and for other currents. We are able to understand how electrons essentially differ from neutrinos and quarks. Thus in the wave with spin $1/2$ there are some elements of physical reality rather than mere probabilities. The quantum wave is not reduced to mere amplitude and phase.

Certainly a large part of quantum mechanics is reduced to these, that part in which **the** phase (thought to be single), which always means the angle of the electric gauge associated to the probability current, is dominant and overrides all other currents.

Even in this case – meaning in the realm of QFT, which is indeed vast because most fermions have an electric charge, but which does not encompass all physics – the quantum wave follows an equation with partial derivatives just as deterministic and relativistic as Einstein's gravitational equations. It is the extended relativistic invariance which gives the quantization of kinetic momentum and this explains Heisenberg's inequalities, which means the limitation of our knowledge about position–energy–momentum (in space-time). Moreover, among principles that may be consequently derived from properties of quantum waves is the exclusion principle expounded by Pauli. This principle states that the occupation number of an electron wave can only be either 0 or 1. The "probability of presence" concept is thus non-verifiable: the experimental validation of any probabilistic law is necessarily made via the convergence of statistical frequencies onto the probability law. And statistics is impossible with only one object. Statistics based on n electrons also includes n electron waves. The probability that in a domain of space D the "electron–particle" is present in D and nowhere else is always calculable but not statistically verifiable from the wave of this lone electron. The situation is absolutely different for a photon because an electromagnetic wave may accommodate myriads of photons. The spatial density of these photons on the wave is proportional to the magnitude of the electric field; this is statistically verifiable.

The concept of probability has two kinds of justifications, *a priori* or *a posteriori*. The concept of probability *a priori*, theorized by Kolmogorov's axioms, defines probability as an additive measure on a family of sets such that the probability of the whole is 1. It is this kind of probability that we encounter in the present book. On the contrary, the concept of *a posteriori* probability is based on randomness, which means an intervention of causes of which we know nothing: For instance a uranium-238 atom exists since its creation, billions of years ago, but suddenly a nucleus of helium is ejected and the nucleus transforms into a thorium-234 nucleus. We do not know what happens, how this process begins, or how it evolves. We only know the end of the process, when the two nuclei separate. Statistics that physicists carry out on an enormous number of uranium-238 atoms allows them to establish probabilistic laws. The probability of decay is constant in time. Its half-life, the duration such that only half of the uranium remains, is 4.4688 billion years. Can this probability be linked to the wave of the protons, neutrons and electrons of this kind of atom? We do not know. And QFT does not know either, despite attempts to causally relate the temporal probability of decay to the nonzero spatial probability of presence beyond the potential barrier. QFT must obviously justify how a spatial probability can yield a temporal probability. This has been discussed by many physicists [101].

Astonishing implications of these probabilities, such as entanglement, Bell's inequalities and Aspect's experiment, are always interpreted with the idea of a quantum wave following Hamiltonian relativistic dynamics. This is in the framework of a theory which replaces the necessary definition of mathematical objects by a set of postulates supposedly universal. But these postulates are not universal, because waves of different fermions of the Standard Model have left and right waves. And never was it proved that these left and right waves would obey the postulates of quantum theory. What we introduce here, using a wave function of space-time with value in $\text{End}(Cl_3)$, can be used to support the mathematical foundations of second quantization. But problems of Hamiltonian dynamics will remain entirely.

Why were we entitled to doubt the possibility of a Hamiltonian relativistic dynamics from the spinor wave of the electron? The problem comes from time, which is revertible in Hamiltonian dynamics and which is not revertible for the invariance under Cl_3^*. Since QFT admits the universal validity of CPT symmetry along with the violation of P-symmetry and of CP-symmetry, this is equivalent to the violation of T-symmetry. It is thus logical to think that the dynamics of fermions in the Standard Model cannot be Hamiltonian. Moreover, we have explained in Chapter 1 how the first Hamiltonian form of the Dirac equation is both nonrelativistic and nonequivalent to the second form of the Dirac equation, which is relativistic. It is thus false to consider two nonequivalent wave equations as describing the same particle!

We now have a much stronger reason, knowing that ordinary time is expressed through the exponential function which applies the Lie algebra Cl_3 on the Cl_3^* group, particularly $\mathbb{R}$ onto $\mathbb{R}^{+*}$: time is oriented by the structure of the whole space-time. The measurements of space and time used in the interpretation of Aspect's experiment and of Bell's inequalities are made in the frame of the space-time of special relativity. All these measurements should account for the inclusion of space-time in the true space-time manifold. We explain in 4.6.2 how events which seem to coincide, as observed in a particular frame, in fact may be in the future of each final observer.

In the previous discussion of the decay of a uranium atom, and with the emission of a photon as well, it is essential to understand the all-or-nothing character of quantum phenomena that is the main feature of quantization. Surely, for any quantum phenomenon the kinetic momentum comes in *integer* multiples of $\hbar/2$. Yet making use of ergodic properties, it is perfectly possible to link temporal probabilities of seemingly random events, to a continuous distribution of spatial probabilities. This is of course something to elaborate on, to account for probabilities used by Einstein for his physics of light.

5.5 The nearly forgotten Dirac equation

We built upon two parts of Einstein's scientific work. The first part of his work was interpreted as the replacement of the invariance group of Newtonian physics laws by another invariance group called the Poincaré group, which is 10-dimensional. Relativistic quantum mechanics has replaced the restricted Lorentz group by the $SL(2,\mathbb{C})$ group. And this group, 6-dimensional, was extended by us to the $GL(2,\mathbb{C}) = Cl_3^*$ group. This extension is justified by the spin $1/2$ of all the fundamental objects of quantum physics: fermions. These are named after Fermi who worked out the statistics indicated by Pauli's exclusion principle. Regarding the exclusion principle, we also went further since it is now linked to the additivity of the fermion mass-energy, through the orthonormalization of the fermion waves. This additivity is not an exact law and is only due to the extremely tiny masses of particles, which makes the nonlinearity of gravitation negligible.

Why was quantum mechanics essentially built from the Schrödinger equation, when only a few months after this first discovery the Dirac equation was also available? De Broglie explained how after the 1927 Solvay Conference, having been appointed professor at the Sorbonne, and aware of the obstacles to his idea of the wave guiding the particle, he began teaching the works of the other quantum physicists, not his own theory. He returned to the ideas of his youth only many years later. Yet a long time prior to this change, he was already interested in the Dirac equation because the equation was relativistic, like his initial concept of a wave associated to the movement of any material particle [53]. But by the time he changed his mind about the explanatory power of quantum mechanics, the Dirac equation was already considered outdated, seldom taught. This area of quantum physics was slowly disappearing from the physics curricula of universities.

Among the reasons for this decline is the great difference introduced by the spin 1/2, between what is called a physical quantity in classical physics and what is called a physical quantity in quantum mechanics. In classical mechanics the quantities are numbers, for instance a temperature of 302 Kelvin. Other quantities are components of vectors like velocity or force, where the components are real numbers. Others, slightly more difficult to understand, are tensors such as the inertia tensor or the electromagnetic tensor. Still, all these quantities are real numbers. The wave equation found by Schrödinger, that of Pauli and more so the Dirac equation, all introduced a deep change: quantum states have no direct link with the quantities of classical physics. To each classical quantity is associated, everywhere in quantum mechanics, an operator acting on the linear space of states; and it is the proper value of this operator which gives the real number of classical physics. According to this line of reasoning, the ultimate physical reality of the electromagnetic field comes from creation and annihilation operators which add or subtract a unity to the number of photons present in the electromagnetic wave.

On the contrary, we explain in Appendix C how all quantum numbers of solutions in the hydrogen case are obtained with only the condition of the normalization of the wave. This does not contradict quantum mechanics, because we may show adequate operators such that each solution is automatically a proper vector of these operators. Nevertheless, the general theory of Hermitian operators is simply useless.

De Broglie remarked early on [54] that with the Dirac wave equation it was still different: certainly the idea of classical numbers as proper values of operators is conserved, but it is not these quantities that have true relativistic variance; it is the tensor densities which transform following the law established for relativistic physics. Several arguments were brought up against the Dirac wave, one of which is that the matrices used in the wave equation are only defined up to an arbitrary matrix factor. It is thus difficult to consider the wave as having any element of physical reality, and the wave appears to be merely a tool for calculations, nonphysical. We resolved this difficulty by defining the Dirac matrices from the Pauli matrices in a unique manner, from the canonical basis of $GL(2,\mathbb{C})$. They are the same for two observers in relative motion, and thus the wave with spin 1/2 may have the status of physical reality, in the same way as for instance an electric field. Some other difficulties are only historical; they were resolved when the study of the tensors in the theory was improved: Hestenes introduced new methods of calculation, much more efficient. They allowed him to prove that the densities of electric charge and of electric current nearly follow the Lorentz force law [1]. Only one other theory derived the laws of motion from field equations: general relativity. This strongly impressed de Broglie when Einstein managed to prove the derivation (de Broglie needed this nonlinearity to link his particle to its wave). And we may say that the improved wave

1. Our improved equation gives the Lorentz force exactly, see 1.9.

equation is even stronger than Einstein's gravitation equations, which gives the law of the movement only for a singularity of the field, while **the wave equation of the electron gives the Lorentz force for any solution of the wave equation**.

Another reason for theoretical physics to discountenance the spin 1/2 wave is that the Dirac equation is only a linear equation. Thus its worth is much less than that of general relativity, which is nonlinear. It is also only a theory for a single electron, and in an exterior potential which is nonsense in a field theory. Yet this criticism applies to the Dirac equation as formulated in 1928, not to our work: the improved equation obtained in Chapter 1 and its subsequent generalizations in the next chapters are nonlinear, both in mass terms and in gauge terms where potentials are dependent on the wave. Algebraic identities suppress the effect of each chiral current on itself. This eliminates the self-effect, without destroying the effect. It is seen only if we consider the entire wave altogether and not merely the different pieces. Furthermore the more useful form of the fermion wave equation is its invariant form, which is not at all linear. The wave is a well-defined function of space-time (not configuration space) with value onto a set of operators acting on themselves. This is the only possible justification for second quantization.

With Lorentz' electron-particle model, the mass-energy is the sum over all space of the energy density of the electromagnetic field. If the electron should be exactly a point, this energy would be infinite. It the electron should be extended, the repulsive force due to the electric field of the charge would be necessarily compensated by other unknown forces. This led us to separately consider the exterior field created by the other charges. In the previous chapters the energy density of the electron was no longer the energy density of the electric field; it was the temporal component of the energy–momentum density linked to the Lagrangian density of the electron. It was previously known that the energy density linked to the electromagnetic field $W = \frac{1}{2}(E^2 + H^2)$ was problematic: the mass of this energy depends on how energy is defined from the mechanical point of view [8]. We see in Chapter 1 that it is the electromagnetic field itself which is the energy–momentum tensor. The mass-energy of the electron is exactly the sum of the energy density of the electron wave. The tensor density of energy–momentum in quantum physics is linked by Noether's theorem to the invariance of the Lagrangian density under space-time translations. Since we only needed the fermion part of the Lagrangian density of the Standard Model, and since wave equations of bosons were derived from those of fermions by the recursion on wave equations, we conclude that we need only the fermion part of the Lagrangian density.

This part of the Lagrangian density is derived from the wave equations, and the wave equations are derived from the Lagrangian density. This suggests that gauge fields have no proper energy. Phenomena where gauge fields seem to own a proper energy are phenomena where it is always possible to reallocate this energy to the fermions that give or receive this energy.

This leads to a first prediction: **As strong as a magnetic field may be around a star (including neutron stars and black holes) or a galaxy, this field, despite its bipolar and multi-polar structure, has absolutely zero effect on the geometry of the gravitational field which can remain perfectly spherical**.

5.6 Why those wave equations and not others?

The global wave equation for all fermions of the first generation separates into 16 equations corresponding to 16 spinors, eight left and eight right, making up the wave. This splitting is what allows us to distinguish each of these objects from others. But the separation is only partial: wave equations are all constructed in the same manner, with a differential part (the only part of the equation that totally distinguishes parts of the wave), a mass term and a gauge term. The mass term and the gauge term contain space-time vectors that are themselves functions of left and right spinors. This dependence of the gauge and mass terms on spinors reveals that the wave equation is highly nonlinear.

We again look at the three parts of our wave equation: The whole equation is constrained by the invariance under the Cl_3^* group that governs the whole of the Standard Model and gravitation. We consider the homothety ratio in terms of the dinum (see 1.7) that we use to distinguish contravariance from covariance.

1. Spinors have a dinum $1/2$. Partial derivatives acting on them give terms with dinum $-1/2$.

2. Thus the other terms must have the same $-1/2$ dinum. And they contain a multiplication by the spinor wave function, with $+1/2$ dinum. Thus the other factors must together bring a -1 dinum. Therefore a single spinor factor is inconvenient, and it is impossible to have quadratic terms with regard to spinors; only cubic terms are possible. These cubic terms bring a supplementary dinum of $+1$, not -1, and thus we have a difference of $+2$ to compensate.

3. This may be done in only two ways, either bringing a -2 dinum or bringing two -1 dinums. The first possibility is what the gauge term brings, where the lone charge (actually g_1, g_2 and g_3 constants) brings a -2 dinum.

4. The second possibility, $-2 = -1 - 1$ is actually what m/ρ brings to the mass term because m brings a -1 dinum and $1/\rho$ also brings a -1 dinum. All in all, there are two, and only two, possible terms in addition to the differential term because there are exactly two possibilities to express 2 as an ordered sum of integers: $0 + 2$ and $1 + 1$. Moreover this justifies the difference between mass and charge, which certainly give both potentials in $1/r$. They are different only from the point of view of the extended invariance.

Why do we not obtain derivatives of higher order? This is because the wave equation Dirac envisioned must have similar partial derivatives for time and space coordinates: this is required by special relativity. And it is necessary to only have first-order derivatives so as to obtain a conservative probability density. First-order derivatives are also the terms of the first approximation. In the study of manifolds, by distinguishing the variation of points and the variation of a mobile basis, it is possible to avoid the writing of differential terms with higher degrees. It is similar to the systems of first-order equations that are obtained in mechanics (where second-order derivatives are natural) when velocities are used as auxiliary variables. Hence the use of only first-order derivatives does not restrict the generality of our wave equations. Furthermore the recursion takes place in the wave equations. Second-order derivatives allow us the definition of the gauge bosons. And similarly, terms of higher order are included in the relations linking gauge fields to potentials, and currents to gauge fields. Lastly the system is closed for another reason: the null dinum of all gauge fields. Consequently, by multiplying operators acting on these gauge fields we still obtain such an operator.

The quantum wave gives two connections on the space-time manifold: a connection linked to the currents of the quantum wave (inertia) and another linked to its invariance group (geometry)(see chapter 4). The identity of these connections is exactly the equivalence principle between inertia and gravitation. And the reason for this identity is: the space-time manifold is a hypersurface of dimension four itself included in the 8-dimensional Cl_3^* Lie group, which is also a manifold.[2] Since the identity directly concerns the Lagrangian density, it also concerns the energy–momentum. The proper mass of quantum wave equations is thus a difference between inertial and gravitational mass, though not noticeable because Avogadro's number is too high. This mass is not defined by the particle alone; it is proper to the particle interacting with a material system great enough to allow measurements.

5.7 Treasure hunt

In the vast "treasure hunt" that is scientific research, it is very easy to let oneself be rerouted by coincidences, the main reason for believing we are following the right track when actually the track is already lost. And there have been several coincidences throughout the history of physics. For instance the wave equation of the electron was discovered at just the same time as the spin 1/2, and at the time, there was yet no direct relation. Another coincidence concerns mathematical tools: the Clifford algebra of 3-dimensional space is also the algebra of complex 2×2 matrices (but only as algebras on the real field!). This serves to justify the habit of quantum

2. The equivalence between the identity of the two connections to the equivalence principle is a true logical equivalence, with double logical implications.

mechanics to only use functions with value in the complex field. A third coincidence: the Lie group of rotations in 3-dimensional space has the same Lie algebra as the group of the 2×2 unitary complex matrices, denoted as $SU(2)$ (but the groups themselves are different!). This gives an additional justification to the sole use of functions with complex values, and the primacy of unitary transformations that conserve the probability. These coincidences, which are accidental from the mathematical point of view, are reasons that led physicists to consider the theory of operators on quantum states as a tool that is all at once necessary, sufficient, and impregnable – but all the while: a false track!

The human spirit always tries to reduce novelties down to what is already known: it's our nature. There still are some people today who persist in restraining the study of electromagnetism to absolute time, which is only time as perceived by our internal biological clock. In the same manner the concept of spinors was systematically reinterpreted, distorted, in order to reduce the new concept to something previously known: tensor physics. Therefore the novelty of the situation was not received, like the infinite kinds of tensor densities that may be constructed from spinor waves. Similarly, only tensor densities which are invariant under the electric gauge have been considered, as if the electron could not also be affected by weak interactions.

5.8 Physics and mathematics

Mathematics and physics are closely related sciences, both concerned with quantifying data and integrating them into an orderly body of knowledge. But these two sciences, both extensively developed, are nowadays so vast that it is impossible for a young scientist to master the whole of physics or the whole of mathematics and even more to master both domains fully.

Galileo pointed out that the language of physics was mathematics, and since then, the connection between physics and mathematics has grown ever closer. But misunderstandings have significantly gotten worse since the beginning of quantum physics. These misunderstandings, as is often the case, can in part be ascribed to both parties. The evolution of mathematics towards greater abstraction and generality is natural but ill-adapted to physicists' needs: the theory of linear spaces is naturally made with a general n-dimensional space, but what is interesting for physics is the just 3-dimensional space and the $3+1$-dimensional space-time. Only with three dimensions does a cross-product exist, which is so useful in physics. Properties specific to a 3-dimensional space (scalar and cross products, curl, and also the electromagnetic field as a field of energy–momentum densities) do not interest most mathematicians. The particular properties of the algebra of the 2×2 Pauli matrices, like the fact that the co-matrices are complex numbers, act only if $n = 2$. Hence the use of general n-dimensional linear spaces, so natural in mathematics, is in practice detrimental for physics.

Physicists are also partly responsible for these misunderstandings. It is impossible to take advantage of the strength of mathematical results when their constraints are disregarded: for instance, the necessary definition of mathematical objects for which the reasoning can be applied, and the importance of theorems of existence and of impossibility. [3] So it is the Pauli algebra, 8-dimensional on $\mathbb{R}$, which is important, while anything else seems to lead the Dirac theory to resort to the use of $M_4(\mathbb{C})$ or the subalgebras $Cl_{1,3}$ and $Cl_{3,1}$. It is also necessary to distinguish similar yet nevertheless different concepts such as a Lie group and a Lie algebra. Quantum theory was built on the basis of mathematics of the century that preceded its beginnings. For instance the concept of function was purely computational, and the questions of limit, of topology and even mere concern for the set of departure and the set of values, or the usefulness of Clifford algebras: all this was misunderstood by physicists who were old enough, despite their remarkable youth, to know only the mathematics of the nineteenth century. Physicists were even misled by the power of distribution theory, which has made the use of Fourier and Laplace transformations much more efficient. Thus when QFT was introduced, most people were quite sure that the needed theorems would necessarily be soon demonstrated. But the expected proofs never came: physicists had too much confidence in the power of mathematics.

5.9 Understanding and predicting

Among the ideas that we now understand better, several were known for a long time. The existence of Planck's constant in physics is more than a century old, and the quantization of kinetic momentum arises from there. Here we also explain how this constant ratio between energy and frequency comes from the invariance of quantum laws under Cl_3^*. This invariance emerges from the mathematical structure of the fermionic waves. And the mathematical structure of the fermion waves in turn comes from this invariance [4]. This is manifested in the equivalence between two forms of the wave equations, due to the invertible character of each value of the wave function. Furthermore this equivalence gives rise to the Lagrangian mechanism. It is an extremal principle: Noether's theorem associates the

3. In his second book on the Dirac theory ([57] Chapter II, section 2) de Broglie clearly explained the following impossibility: if we consider three operators m_x, m_y and m_z satisfying anti-commutation relations of 3-dimensional rotations, all possible proper values of m_z are $-j, -j+1, \dots, j-1, j$ where $j(j+1)$ is proper value of $m_x^2+m_y^2+m_z^2$, and all possible values of j are $0, 1/2, 1, 3/2, 2, 5/2 \dots$ But if m_x, m_y and m_z are angular momentum operators ($m_x = i(y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y})$ and so on), the only possible values of j are $0, 1, 2, 3, \dots$. Consequently the operators of the Dirac theory, with values $j = 1/2, 3/2, 5/2, \dots$ are not angular momentum operators! (Thus we named these operators as "kinetic momentum" operators).

4. This looping causality is the only reason explaining why metaphysics is unnecessary.

translational invariance of wave equations to the existence of conservative densities of energy–momentum. This theorem also associates a conservative kinetic momentum tensor to the invariance under Cl_3^*. We consequently obtain the quantization of kinetic momentum with the expected value $\hbar/2$. And since the kinetic momentum is quantized, the Planck constant appears fixed. The orthonormalization of the wave and the resulting quantization of the kinetic momentum justify the use of kinetic momentum operators that give the different states of electrons in atoms.

Pauli's exclusion principle has also been known for nearly a century. We relate this principle to the necessary orthonormalization of states in the case of the electron in an atom. This orthonormalization is itself related to the additivity of the mass-energy, and hence to the properties of gravitational sources, since masses of microphysical objects are very small in comparison with the masses necessary to reveal the nonlinear character of gravitation. So the energies of the various electrons in an atom are additive. This additivity of the energy is itself related to the additivity of the gauge potentials. It is enough to justify that the gravity around a star is proportional to the total mass of the star, the sum of the masses of all its components. This is only a linear approximation, legitimate in the case of a low gravitational field.

We also linked the equivalence principle to this weak-field approximation, through Lagrangian densities that may be written as the difference of an inertial part and a gravitational part. Noether's theorem hence gives two equal energy–momentum tensors, which thus have the same temporal component. By integrating over space we thus obtain the equality between inertial mass and gravitational mass. This was the starting point of Einstein's general relativity.

The inclusion of the space-time manifold into the Lie group Cl_3^* explains the homogeneity and isotropy of our space-time: in a Lie group, the vicinity of any element of the group is similar to the vicinity of the neutral element. This is largely established from the experimental point of view, where the cosmic background radiation is still today very close to homogeneity (one part in 10^5). Moreover the geometry of the physical space naturally tends to infinity (like that of $\mathbb{R}^3$) if the relative manifold of the x is linked to the invariant manifold of X by $\mathrm{x} = \phi X \phi^\dagger$.

5.10 Falsifiability

Any scientific theory must be falsifiable: it should be possible to prove that the theory is false. Conversely it is impossible to prove *definitively* that the theory is true. Hence this can only humble the authors of this work. Will the best theory someday completely do without the Dirac equation in understanding the properties of electrons, neutrinos, quarks and other "particles"? Even if the Dirac equation gives all known results for electrons in atoms, we proved that it is possible to get the same results from another point of departure [34].

From Fermat's principle through to Lagrangian mechanics and up to the Standard Model, the whole of physical theory has been developed from an extremal principle. Is this principle fundamental in physics? The answer that we gave in 2.3.4 is clearly: No! We detailed how the algebraic structure of Cl_3^* gives the double logical link between the wave equation and Lagrangian density. Thus the extremal principle is not fundamental, though it is very efficient because the invariance of a Lagrangian density gives, through Noether's theorem, conservative quantities. And that which conserves, which is stable, is much easier to understand than that which is furtive, unstable, changing, unpredictable. Furthermore the extremal principle is the reason for the unity of all matter-energy, because each fermion contributes to this energy–momentum tensor, whose temporal component gives the energy of matter. In addition, the electromagnetic field itself is this energy–momentum. Moreover only fermions contribute; photons only transport the energy–momentum between two fermions. Nevertheless in a regime dominated by gravitation there is no longer a Lagrangian formalism (see 4.3) and thus no laws of conservation.

The great debate of quantum physics was around the question: what *is* the quantum object? A particle? (A very small object, even an infinitely small point?) A wave? A wave *and* a particle, as de Broglie thought? Any phenomenon in quantum physics that is adequately described with particles can also be adequately described with waves, and conversely. And it is also possible to describe the same phenomenon with objects that are both waves and particles [94]. Here we began from the Dirac wave. And we even claimed: an electron is an orthonormalized quantum wave. Is the electron also a point object? Nothing forbids this! It is possible that our orthonormalized quantum waves may include singularities, or even must include singularities. To ascertain this it will be necessary to solve the wave equations, study the solutions carefully, and understand in particular the emission and absorption of photons and of the other bosons. Note also that the solutions calculated, among which are our solutions for an electron in a hydrogen atom, may be qualified as "solitons": the appearance of the radial functions means that these solutions are similar to solitons. Their linear approximations are completely stable, definitely. Their "loneliness" is simply less visible because these waves are not separated in ordinary space, but orthogonal for a scalar product concerning the Cl_3^* manifold where our space-time is only a 4-dimensional submanifold.

The great debate began at a time when only one elementary particle was really known: the electron. Moreover the wave of the electron is not elementary but double, made of a left and a right wave. Other particles known in the early years of nuclear physics – protons and neutrons – are no longer considered elementary since they are made of three quarks.

We are very far from a complete exploration of all the consequences of the extension of the invariance group. Consider once more a common phenomenon like the transition of an electron of the solar photosphere from one energy state to another, followed by the travel of a photon to our eye and its absorption by an electron in our retina. We interpret this chain of events by attributing an electromagnetic wave to the photon. This allows us to neglect both the emitting electron and the receiving electron during the transport. But the duration of this transport, from the point of view of the photon, is exactly null. We might as well describe the event as follows: the electron wave in the photosphere produces a energy–momentum tensor which is also the electromagnetic field that propagates towards an electron in our retina, giving a direct interaction of two fermion waves, that of an electron in the Sun and that of an electron in the retina.

A great number of things remain to be understood. To give the wave equations of the quarks is only a first step. It will also be necessary to know how to calculate magnetic moments of the proton and the neutron, and to study what new equations would allow nuclear physicists to understand. And it may happen that many other consequences exist which we did not even think of.

What we already obtained fully justifies the extension of the invariance group from $SL(2, \mathbb{C})$ into Cl_3^*. **Without this extension there is neither quantization of the kinetic momentum, nor the double presence of the chiral invariance both in the gauge symmetries and in the geometry of gravitation. Without this extension physics can understand neither the reason for the existence of the neutrino–monopole, nor the values for the electric charge of leptons and quarks of each generation of the Standard Model, nor the specificity of gauge fields**: their dinum is null, they are sensible only to the part containing Lorentz transformations of the similitude group. The products of such fields also have a null dinum, and themselves behave as gauge fields: this makes possible the construction of creation and annihilation operators.

In metrology, physicists are nowadays working to replace the old Standard Kilogram by this standard of action which is the Planck constant. This is perfectly compatible with the extended invariance: when a similitude multiplies all lengths by r, the length of all standard meters are also multiplied by r. Thus the physicist who always locally measures only the ratio between the length of the measured object and the length of the standard meter cannot see the homothety. We may say the same thing for proper masses or for actions, replacing only r with r^3 and r^4 respectively.

Epilogue

Thank you for your patience! This book is the result of thirty years of hard work. We present the following as a recap of the most salient features of this work. We introduced five novelties:

1. The natural mathematical framework of the quantum wave is the Cl_3 algebra (instead of space-time algebra); it is enough to describe both the quantum wave and the space-time manifold.

2. The invariance group (form invariance which in quantum physics replaces the Lorentz group of special relativity), is extended from $SL(2,\mathbb{C})$ to the $GL(2,\mathbb{C}) = Cl_3^*$ group.

3. The linear Dirac equation is replaced by our improved (and nonlinear) wave equation, obtained by simplifying the mass term of the Lagrangian density.

4. Space-time is not the starting point, but it is a consequence of the field of values of the fermion waves.

5. The space-time manifold is included in Cl_3^* (invariance group of all physical laws) as its auto-adjoint part.

The calculations with Cl_3 are much simpler than those with 4×4 complex matrices. The first yield of these simplifications is a better understanding:

1. Why there is an extremal principle in quantum mechanics.

2. Why there is an equivalence principle in general relativity.

3. Why there is the double equality $E = mc^2 = h\nu$ (Einstein's relation and existence of the quantum of action).

4. Why there is the quantization of electric charges and action.

5. Why there is the maximal violation of parity in weak interactions.

6. Why there is the spin 1/2 and not integer values for angular momentum operators.

7. Where the exclusion principle comes from.

8. Why this kind of wave equations and how these equations can be linked to the geometry of space-time.

9. What charge conjugation is and how the puzzle of the negative energies is solved.

10. How the fermion part and the boson part of the standard model are connected.

11. Why the baryonic number is conserved.

12. Why all leptons are insensitive to strong interactions.

All these results are obtained without any metaphysical principle. By enlarging the linear space of values of the quantum wave to $Cl_{3,3}$, the fermion wave integrates most of the novelties introduced by the Standard Model:

1. The existence of exactly two quarks with three color states, for each generation.

2. The linked existence of three generations and of charge conjugation.

3. The gauge invariance under the $U(1) \times SU(2) \times SU(3)$ group of the Standard Model, and the impossibility of a greater gauge group.

4. The distinction between leptons insensitive to strong interactions and quarks linked by strong interactions with color. For the lepton sector the existence of one particle with an electric charge and of one magnetic charge, the magnetic-monopole–neutrino with a total of four waves linked to the four kinds of representations of the Cl_3^* group. The neutrino with a right and a left wave, and Lochak's magnetic monopole are the same object.

5. For quarks the existence of twelve elementary waves, three for each of the four kinds of representations of Cl_3^*, with six of these waves, three left and three right, forming the three quarks of a proton or of a neutron.

6. The quantization of kinetic momentum with the value $\hbar/2$ for each elementary particle (it is precisely for this reason that they may be considered particles), namely: the proton, neutron, electron and neutrino. This explains the confinement of the quarks in protons and neutrons.

7. The magnitude of the electric charges of all particles (electrons, neutrinos, quarks...) and of their antiparticles.

8. The origin of the preference for left waves (see 3.8). The inclusion of Cl_3 into $\mathrm{End}(Cl_3)$ also explains why the electron wave in second quantization can account for all results given by its wave of first quantization.

9. About the geometry of space-time, we also resolve the ambiguity of the signature of space-time in special relativity. Since the quadratic form giving the space-time metric comes from the determinant in (1.31), the signature is necessarily $+ - - -$.

10. The equivalence between wave equations in the usual form and wave equations in the completely invariant form requires the cancellation of the X term in 4.3.

11. The existence in the electromagnetic field of quanta of energy–momentum (photons), because the electromagnetic field is made of components of the energy–momentum densities of the fermion field.

12. The existence in quantum physics of a probability density and the necessity to normalize the quantum wave. This results from the equivalence principle between gravitation and inertia.

The inclusion of the space-time manifold into the Cl_3^* Lie group brings:

1. The geometric origin of the arrow of time.

2. A better understanding of non-simultaneity in optics.

3. A mainly geometric origin for the expansion of the universe, and its recent acceleration.

All this work could not have even begun without the creation, by Louis de Broglie himself, of a free foundation with the aim of continuing his physics research. The head of this "Fondation Louis de Broglie" was Georges Lochak (1931-2021), who discovered the leptonic magnetic monopole, which is the starting point of our work.

If you have questions or comments you may use our email addresses.

The path is arduous, but Louis de Broglie declared the necessity of both freedom and imagination "**Pour l'avenir**."

Appendix A

Clifford algebras

We present what a Clifford algebra is. We study the algebra of the Euclidean plane and the algebra of 3-dimensional space. This algebra is also generated by Pauli matrices. We include here space-time and relativistic invariance. We study different tensor densities of the electron wave. We prove identities necessary to obtain the form invariance. We study left and right currents, potential vectors and the electromagnetic field.

Clifford algebra is a useful tool: the physics of light, of gravitation, and quantum physics need waves; thus they need trigonometric functions. Trigonometry is highly simplified with the use of the exponential function. This exponential function needs a product. The addition of vectors is not enough, a multiplication is necessary: we must be able to use both addition and multiplication of vectors. Mathematics provides the structure of algebra. Here we present this algebra at a level of minimal difficulty. As this is a presentation for physics needs, we expect our pedagogical decision to be met with some criticism from mathematicians. For instance we choose to speak only about real Clifford algebras, though algebras on the complex field also exist. We might think that they ought to be essential in quantum physics since the most frequently used Clifford algebra is also a complex algebra. But it is actually its structure as a real algebra which is useful in quantum physics [1]. We may also consider that it is not the algebra which is important but only the ring structure, and even only the multiplicative Lie group structure. The presentation here is intentionally made for beginners, not for theorists of Lie groups.

1. A real Clifford algebra has vectors whose components are real numbers and which are never multiplied by i. A complex Clifford algebra has vectors whose components are complex numbers which may be multiplied by i. You can also refer to Doran and Lasenby [63], which is more oriented towards space-time algebra.

A.1 What is a Clifford algebra?

1. It is an algebra [9][23], and there are two operations: an addition denoted by $A + B$ and a multiplication denoted by AB, such that for any A, B, C:

$$\begin{aligned}
&A + (B + C) = (A + B) + C \;\; ; \;\; A + B = B + A, \\
&A + 0 = A \;\; ; \;\; A + (-A) = 0, \\
&A(B + C) = AB + AC \;\; ; \;\; (A + B)C = AC + BC, \\
&A(BC) = (AB)C.
\end{aligned} \tag{A.1}$$

This last equality (associativity of multiplication) allows us to suppress parentheses. The product is thus simply denoted as ABC.

2. The algebra contains a set of vectors, denoted with arrows, in which a scalar product exists and the Clifford multiplication $\vec{u}\vec{v}$ is supposed to satisfy for any vector $\vec{u}$ the identity:

$$\vec{u}\vec{u} = \vec{u} \cdot \vec{u}. \tag{A.2}$$

where $\vec{u} \cdot \vec{v}$ points out the scalar product [2] of these vectors. This implies, since $\vec{u} \cdot \vec{u}$ is a real number, that the algebra which contains the vectors also contains the real numbers.

3. Real numbers commute with any member of the algebra: if a is a real number and if A is any element in the algebra:

$$aA = Aa, \tag{A.6}$$

$$1A = A. \tag{A.7}$$

These algebras exist for any finite-dimensional linear space. The smallest algebra is unique, up to an isomorphism.

Relations (A.1) and (A.7) imply that the algebra is also a linear space which must be distinguished from the initial linear space. If we start from an n-dimensional linear space the dimension of the algebra is 2^n. We will see for instance in A.3 that the algebra of the usual space, 3-dimensional, is an 8-dimensional linear space.

It is useless to distinguish left and right linear spaces, because real numbers commute with anything. It is also needless to consider the multiplication by a real number as a third operation since it is a particular case of the Clifford multiplication.

2. The scalar product satisfies, for any vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and any real number a:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}, \tag{A.3}$$

$$(a\vec{u}) \cdot \vec{v} = a(\vec{u} \cdot \vec{v}), \tag{A.4}$$

$$(\vec{u} + \vec{v}) \cdot \vec{w} = (\vec{u} \cdot \vec{w}) + (\vec{v} \cdot \vec{w}). \tag{A.5}$$

We recall also that the scalar product of two vectors is the product of the lengths of these vectors by the cosine of the angle that they form.

If $\vec{u}$ and $\vec{v}$ are two orthogonal vectors (this means if $\vec{u}\cdot\vec{v}=0$) the equality $(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})=(\vec{u}+\vec{v})(\vec{u}+\vec{v})$ implies

$$\vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v}=\vec{u}\vec{u}+\vec{u}\vec{v}+\vec{v}\vec{u}+\vec{v}\vec{v},$$

thus we have:

$$0=\vec{u}\vec{v}+\vec{v}\vec{u} \quad ; \quad \vec{v}\vec{u}=-\vec{u}\vec{v}. \tag{A.8}$$

This is the major difference compared to usual rules of calculation with numbers: the multiplication is not commutative and anyone must be as cautious as for matrix calculus. It is moreover always possible to perform all calculations using an algebra of squared matrices. The addition is defined in the whole algebra, which contains both numbers and vectors. Then we will get sums of numbers and vectors: $3+5\vec{i}$ is **authorized**. This may perhaps seem strange or disturbing, but it is not any different from $3+5i$ and everyone using complex numbers finally gets used to it. The two following definitions are important and general:

Even subalgebra: The even subalgebra is the subalgebra generated by all products of an even number of vectors: : $\vec{u}\vec{v}$, $\vec{e}_1\vec{e}_2\vec{e}_3\vec{e}_4$, and so on.

Reversion: The reversion $A\mapsto\widetilde{A}$ changes the order of products. Reversion does not change numbers a nor vectors: $\tilde{a}=a$, $\widetilde{\vec{u}}=\vec{u}$, and we get, for any $\vec{u}$ and $\vec{v}$, or A and B:

$$\widetilde{\vec{u}\vec{v}}=\vec{v}\vec{u} \ ; \quad \widetilde{AB}=\tilde{B}\tilde{A} \ ; \quad \widetilde{A+B}=\tilde{A}+\tilde{B}. \tag{A.9}$$

A.2 Clifford algebra of a Euclidean plane

The algebra of the Euclidean plane Cl_2 contains all real numbers and all vectors of an Euclidean plane, $\vec{u}=x\vec{e}_1+y\vec{e}_2$, where $\vec{e}_1$ and $\vec{e}_2$ form a direct orthonormal basis of the plane: this means that they are two vectors with length 1, orthogonal to each other; they satisfy: $\vec{e}_1{}^2=\vec{e}_2{}^2=1$, $\vec{e}_1\cdot\vec{e}_2=0$. Usually we let: $i:=\vec{e}_1\vec{e}_2$. The general element of the algebra of the plane is expressed as:

$$A=a+x\vec{e}_1+y\vec{e}_2+b\vec{e}_1\vec{e}_2=a+x\vec{e}_1+y\vec{e}_2+ib, \tag{A.10}$$

where a, x, y and b are real numbers. This is enough because:

$$\begin{aligned}
\vec{e}_1 i &= \vec{e}_1(\vec{e}_1\vec{e}_2)=(\vec{e}_1\vec{e}_1)\vec{e}_2=1\vec{e}_2=\vec{e}_2,\\
\vec{e}_2 i &= -\vec{e}_1 \quad ; \quad i\vec{e}_2=\vec{e}_1 \quad ; \quad i\vec{e}_1=-\vec{e}_2,\\
i^2 &= ii=i(\vec{e}_1\vec{e}_2)=(i\vec{e}_1)\vec{e}_2=-\vec{e}_2\vec{e}_2=-1.
\end{aligned} \tag{A.11}$$

We have two remarks:

1. The even subalgebra Cl_2^+ is the set formed by all $a+ib$; thus it is the complex field. This subalgebra is commutative. We can say that complex numbers are underlying as soon as the dimension of the linear space is greater than one.

2. Here the reversion is the complex conjugate $\tilde{i} = \widetilde{\vec{e}_1\vec{e}_2} = \vec{e}_2\vec{e}_1 = -i$

We thus obtain, for any $\vec{u}$ and any $\vec{v}$ in the plane: $\vec{u}\vec{v} = \vec{u}\cdot\vec{v} + i\det(\vec{u},\vec{v})$ where $\det(\vec{u},\vec{v})$ is the determinant of two vectors in the $(\vec{e}_1,\ \vec{e}_2)$ basis.

To establish that $(\vec{u}\cdot\vec{v})^2 + [\det(\vec{u},\vec{v})]^2 = \vec{u}^{\,2}\vec{v}^{\,2}$, it is possible to use $\vec{u}\vec{v}\vec{v}\vec{u}$ which may be calculated in two ways, and we recall that $\vec{v}\vec{v}$ is a real number, which thus commutes with anything in the algebra.

A.3 Clifford algebra of 3-dimensional space

The dimension 3 of the usual space is the main reason for the significance of this algebra. We explain in Chapter 1 why other reasons exist for preferring this framework for quantum physics.

The algebra, denoted as Cl_3, contains [3] all real numbers and all vectors of the geometry of space which read:

$$\vec{u} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 =: x^j\vec{e}_j, \tag{A.12}$$

where x^1, x^2, x^3 are real numbers and $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ form an orthonormal basis. The second equality is the usual Einstein summation convention, with Latin indices from 1 to 3. The scalar product satisfies:

$$\vec{e}_1\cdot\vec{e}_2 = \vec{e}_2\cdot\vec{e}_3 = \vec{e}_3\cdot\vec{e}_1 = 0 \ \ ; \ \ \vec{e}_1^{\ 2} = \vec{e}_2^{\ 2} = \vec{e}_3^{\ 2} = 1. \tag{A.13}$$

We let:

$$i_1 = \vec{e}_2\vec{e}_3 \ \ ; \ \ i_2 = \vec{e}_3\vec{e}_1 \ \ ; \ \ i_3 = \vec{e}_1\vec{e}_2 \ \ ; \ \ i = \vec{e}_1\vec{e}_2\vec{e}_3. \tag{A.14}$$

This gives:

$$i_1^2 = i_2^2 = i_3^2 = i^2 = -1, \tag{A.15}$$

$$i\vec{u} = \vec{u}i \ \ ; \ \ i\vec{e}_j = i_j \ ,\ j = 1,\ 2,\ 3. \tag{A.16}$$

In the calculation of squares we may use the method of (A.11). To obtain the commutation of i with all vectors we may begin to prove that i commutes with each $\vec{e}_j$. General element in Cl_3 is: $A = a + \vec{u} + i\vec{v} + ib$. For Cl_3 this gives $1+3+3+1 = 8 = 2^3$ dimensions. We have five remarks:

1. The center of Cl_3 is the set of the $a+ib$ terms. They are the only elements commuting with all the others in the algebra. It is the complex field: $\mathbb{C}$. This is the main reason for the important role of complex numbers in quantum physics. If n is even, the center of a real Clifford algebra is only the real field. The larger center, in Cl_3, has many consequences.

2. The even subalgebra Cl_3^+ is the set of $a + i\vec{v}$ which is isomorphic to the quaternion field. Using the quaternion field we automatically use the Cl_3 algebra which is sometimes called algebra of the biquaternions.

3. $\widetilde{A} = a + \vec{u} - i\vec{v} - ib$: The reversion is the conjugation for the complex field and also for quaternions in Cl_3^+.

4. The $i\vec{v}$ term is usually called an axial vector or pseudovector, while $\vec{u}$ is the true vector or (in short) vector. It is well known that this situation is proper to the dimension 3.

5. Four different and independent terms with square -1 exist: four different ways to get complex numbers. Nonrelativistic quantum theory which always uses a unique term with square -1, is analogous to plane geometry (2D software). What physics actually needs is 3-dimensional space and thus Cl_3 algebra (3D software).

A.3.1 Vector product, orientation

Reminder: Given vectors $\vec{u}$ and $\vec{v}$, the vector product $\vec{u}\times\vec{v}$ is the vector orthogonal to $\vec{u}$ and to $\vec{v}$, with a length equal to the product of the lengths of $\vec{u}$ and $\vec{v}$ by the sine of their angle, and such that the basis $(\vec{u},\ \vec{v},\ \vec{u}\times\vec{v})$ is direct. Using coordinates in the basis $(\vec{e}_1,\ \vec{e}_2,\ \vec{e}_3)$, it is easy to prove, for any $\vec{u}$ and $\vec{v}$:

$$\vec{u}\vec{v} = \vec{u}\cdot\vec{v} + i\ \vec{u}\times\vec{v}, \tag{A.17}$$

$$(\vec{u}\cdot\vec{v})^2 + (\vec{u}\times\vec{v})^2 = \vec{u}^{\,2}\vec{v}^{\,2}. \tag{A.18}$$

From (A.17) we deduce:

$$\vec{u}\cdot\vec{v} = \frac{1}{2}(\vec{u}\vec{v}+\vec{v}\vec{u});\ \ \vec{u}\times\vec{v} = \frac{1}{2i}(\vec{u}\vec{v}-\vec{v}\vec{u}). \tag{A.19}$$

Dividing (A.18) by the right term, and taking θ to be a measure of the angle $(\vec{u},\vec{v})$, we get:

$$1 = \frac{(\vec{u}\cdot\vec{v})^2}{(\parallel\vec{u}\parallel\,\parallel\vec{v}\parallel)^2} + \frac{(\vec{u}\times\vec{v})^2}{(\parallel\vec{u}\parallel\,\parallel\vec{v}\parallel)^2} = \cos^2(\theta) + \Big(\frac{\vec{u}\times\vec{v}}{\parallel\vec{u}\parallel\,\parallel\vec{v}\parallel}\Big)^2,$$

$$\frac{\vec{u}\times\vec{v}}{\parallel\vec{u}\parallel\,\parallel\vec{v}\parallel} = \sqrt{1-\cos^2(\theta)} = |\sin(\theta)|, \tag{A.20}$$

$$\parallel\vec{u}\times\vec{v}\parallel = \parallel\vec{u}\parallel\,\parallel\vec{v}\parallel\ \ |\sin(\theta)|. \tag{A.21}$$

Next $\det(\vec{u},\vec{v},\vec{w})$ refers to the determinant whose columns contain the coordinates of the vectors $\vec{u}$, $\vec{v}$, $\vec{w}$, in the basis $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$. Using these coordinates again it is possible to prove, for any $\vec{u}$, $\vec{v}$, $\vec{w}$:

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \det(\vec{u},\vec{v},\vec{w}), \tag{A.22}$$

$$\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{w}\cdot\vec{u})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}, \tag{A.23}$$

$$\vec{u}\vec{v}\vec{w} = i\det(\vec{u},\vec{v},\vec{w}) + (\vec{v}\cdot\vec{w})\vec{u} - (\vec{w}\cdot\vec{u})\vec{v} + (\vec{u}\cdot\vec{v})\vec{w}. \tag{A.24}$$

From the mixed product (A.22) we deduce that $\vec{u}\times\vec{v}$ is orthogonal to $\vec{u}$ and to $\vec{v}$. The determinant (A.22) gives the orientation. We recall that a basis $(\vec{u},\vec{v},\vec{w})$ is said to be direct (to have the same orientation as $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$) if $\det(\vec{u},\vec{v},\vec{w})>0$, and is said to be inverse (to have the contrary orientation) if $\det(\vec{u},\vec{v},\vec{w})<0$. The rule (A.24) allows us to establish that, $B=(\vec{u},\vec{v},\vec{w})$ being any orthonormal basis, then $\vec{u}\vec{v}\vec{w}=i$ if and only if B is direct, and $\vec{u}\vec{v}\vec{w}=-i$ if and only if B is inverse. In the case where $\vec{u}\vec{v}\vec{w}=i$ we also have:

$$\vec{w}=\vec{u}\times\vec{v};\ \vec{u}=\vec{v}\times\vec{w};\ \vec{v}=\vec{w}\times\vec{u}. \tag{A.25}$$

On the contrary, with the other orientation where $\vec{u}\vec{v}\vec{w}=-i$, we have:

$$\vec{w}=\vec{v}\times\vec{u};\ \vec{u}=\vec{w}\times\vec{v};\ \vec{v}=\vec{u}\times\vec{w}. \tag{A.26}$$

Therefore i is fully linked to the orientation of space. Changing i into $-i$ is equivalent to changing the space orientation. The fact that i determines the space orientation plays an essential role in the physics of magnetism and of weak interactions.

All calculations in Cl_3 result from the sum (where we add numbers to numbers, vectors to vectors and so on) and the product (product of two numbers, product of a number and a vector, product of two or three vectors), through the scalar product, the vector product and the mixed product, all well known to physicists and engineers. In Cl_3 algebra there are no mysteries nor undue complications. This should be taught in any technical university.

A.3.2 Pauli algebra

This algebra, introduced in physics as early as 1926 to account for the spin 1/2 of the electron, is the algebra $M_2(\mathbb{C})$ formed by 2×2 complex matrices. It is equal – isomorphic, to be precise – to Cl_3, but only as an algebra on the real field [3]. Identifying the complex numbers with the scalar matrices, and the basis vectors e_j with the Pauli matrices σ_j is enough to determine this identification [4]. And it is fully compatible with our previous calculations because:

$$\sigma_1\sigma_2\sigma_3=\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix}=i, \tag{A.27}$$

$$\sigma_1\sigma_2=i\sigma_3 \quad ; \quad \sigma_2\sigma_3=i\sigma_1 \quad ; \quad \sigma_3\sigma_1=i\sigma_2. \tag{A.28}$$

3. Pauli algebra has a dimensionality of 8 on the real field, and only 4 on the complex field.

4. The identifying process may be considered a lack of rigor, but in fact it is frequent in mathematics. The same process allows us to include integer numbers into relative numbers, or real numbers into complex numbers. To do without this process results in very complicated notations. This process considers $(\sigma_1,\ \sigma_2,\ \sigma_3)$ as a direct basis in the usual space.

Consequently the reverse is identical to the adjoint (transposed conjugate matrix):

$$\tilde{A} = A^{\dagger} = (A^*)^t. \tag{A.29}$$

We will thus refer to this algebra as either Cl_3 or the Pauli algebra. Some Cliffordians who did not understand the concept of isomorphism refuse to use matrix calculus. As for physicists, who have the habit of using Pauli algebra with complicated older notations, the full use of Cl_3 simply brings simplifications to their calculations, without changing their results.

A.3.3 Three conjugations are useful

$A = a+\vec{u}+i\vec{v}+ib$ is the sum of the even element $A_1 = a+i\vec{v}$ (quaternion) and of the odd part $A_2 = \vec{u}+ib$. We define the P conjugation (called "parity" in quantum physics) such that:

$$P : A \mapsto \widehat{A};\ \widehat{A} = A_1 - A_2 = a - \vec{u} + i\vec{v} - ib. \tag{A.30}$$

For any elements A and B in Cl_3 this conjugation satisfies:

$$\widehat{A+B} = \widehat{A} + \widehat{B} \ \ ; \ \ \widehat{AB} = \widehat{A}\widehat{B}. \tag{A.31}$$

P is the main automorphism of the algebra. Any Clifford algebra possesses a similar involutive (meaning: PP is the identity) automorphism. From this conjugation and from the reversion, we can define a third conjugation:

$$\overline{A} = \widehat{A}^{\dagger} = a - \vec{u} - i\vec{v} + ib \ \ : \ \ \overline{A+B} = \overline{A} + \overline{B} \ \ ; \ \ \overline{AB} = \overline{B}\ \overline{A}. \tag{A.32}$$

The composition, in any order, of two among these three conjugations gives the third one. Only P conserves the order of the products, while $A \mapsto \overline{A}$ and $A \mapsto A^{\dagger}$ inverse the order of the factors. Now a, b, c, d being any complex numbers and $\overline{a} = a^*$ the conjugate complex [5] of a, we can prove that for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the Pauli algebra [6] we have:

$$\widetilde{A} = A^{\dagger} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \ \ ; \ \ \widehat{A} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix} \ \ ; \ \ \overline{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{A.33}$$

$$A\overline{A} = \overline{A}A = \det(A) = ad - bc\ ;\ \ \widehat{A}A^{\dagger} = A^{\dagger}\widehat{A} = [\det(A)]^*;\ A + \overline{A} = \mathrm{tr}(A).$$

If $\det(A) \neq 0$ we then get:

$$[\det(A)]^{-1}\overline{A}A = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ A^{-1} = [\det(A)]^{-1}\overline{A}. \tag{A.34}$$

5. The notation $\overline{a}$ for the conjugate is today the only notation used in mathematics, though the notation a^* was commonly used in course books for quantum physics. Thus we allow ourselves the use of either notation when it is clear that there can be no confusion due to $\overline{A} = \widehat{A}^{\dagger}$.

6. The equality $A\overline{A} = \overline{A}A$ is general in Cl_3. The equality $A\overline{A} = \det(A)$ uses the identification between real numbers and scalar matrices, which means the inclusion of the real numbers in the Clifford algebra.

A.3.4 Gradient, divergence and curl

In Cl_3 we use the differential operator:

$$\vec{\partial} = \vec{e}_1\partial_1 + \vec{e}_2\partial_2 + \vec{e}_3\partial_3 = \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix}, \tag{A.35}$$

with [7] :

$$\vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 \ \ ; \ \ \partial_j = \frac{\partial}{\partial x^j}. \tag{A.36}$$

The Laplacian is simply the square of $\vec{\partial}$:

$$\Delta = (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2 = \vec{\partial}\vec{\partial}. \tag{A.37}$$

When applied to a scalar, $\vec{\partial}a$ is the gradient of a, and when applied to a vector $\vec{u}$ we get both the divergence and the curl:

$$\vec{\partial}a = \vec{\mathrm{grad}}\ a = (\partial_1 a)\sigma_1 + (\partial_2 a)\sigma_2 + (\partial_3 a)\sigma_3, \tag{A.38}$$

$$\vec{\partial}\vec{u} = \vec{\partial}\cdot\vec{u} + i\,\vec{\partial}\times\vec{u};\ \vec{\partial}\cdot\vec{u} = \mathrm{div}\vec{u} = \partial_1 u^1 + \partial_2 u^2 + \partial_3 u^3, \tag{A.39}$$

$$\vec{\partial}\times\vec{u} = \mathrm{curl}(\vec{u}) = (\partial_2 u^3 - \partial_3 u^2)\sigma_1 + (\partial_3 u^1 - \partial_1 u^3)\sigma_2 + (\partial_1 u^2 - \partial_2 u^1)\sigma_3.$$

Thus, for any function with a scalar value $a = a(\vec{x})$ and for any function with a vector value $\vec{v} = \vec{v}(\vec{x})$ we have:

$$\vec{\partial}(\vec{\partial}a) = (\vec{\partial}\vec{\partial})a = \Delta a;\ \vec{\partial}(\vec{\partial}\vec{v}) = (\vec{\partial}\vec{\partial})\vec{v} = \Delta\vec{v}, \tag{A.40}$$

$$\vec{\partial}\cdot(\vec{\partial}\times\vec{v}) = 0;\ \vec{\partial}\times(\vec{\partial}a) = 0, \tag{A.41}$$

$$\vec{\partial}\times(\vec{\partial}\times\vec{v}) = \vec{\partial}(\vec{\partial}\cdot\vec{v}) - \Delta\vec{v}. \tag{A.42}$$

A.3.5 Space-time in the Pauli algebra

This surprising inclusion was made as soon as Dirac derived his wave equation (1928):

$$\mathrm{x} = \mathrm{x}^\mu\sigma_\mu = \begin{pmatrix} \mathrm{x}^0 + \mathrm{x}^3 & \mathrm{x}^1 - i\mathrm{x}^2 \\ \mathrm{x}^1 + i\mathrm{x}^2 & \mathrm{x}^0 - \mathrm{x}^3 \end{pmatrix};\ \sigma_0 = \sigma^0 = I;\ \mathrm{x}^0 = ct, \tag{A.43}$$

where c is the speed of light and t is time. Any element M of the Pauli algebra is sum of a vector v and of the product of a second vector w by i:

$$M = v + iw;\ v = \frac{1}{2}(M + M^\dagger)\ ;\ \ v^\dagger = v \tag{A.44}$$

$$iw = \frac{1}{2}(M - M^\dagger)\ ;\ \ w^\dagger = w. \tag{A.45}$$

7. This operator $\vec{\partial}$ is usually denoted in quantum mechanics as a scalar product, for instance $\vec{\sigma}\cdot\vec{\nabla}$. From there results much confusion. Simple notations are very useful for optimizing calculations.

These two space-time vectors v and w are unique. Since $\mathrm{x} = \mathrm{x}^\dagger$, space-time is the set of $M = v + iw$ such that $w = 0$. We call this set the self-adjoint part of Cl_3. In this framework we need two differential operators:

$$\nabla = \partial_0 - \vec{\partial} \ \ ; \ \ \widehat{\nabla} = \partial_0 + \vec{\partial}. \tag{A.46}$$

They allow us to calculate the D'Alembertian:

$$\nabla\widehat{\nabla} = \widehat{\nabla}\nabla = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2 =: \Box. \tag{A.47}$$

The main reason for the use of Cl_3 is in these equations, which means that the D'Alembertian includes the P transformation $M \mapsto \widehat{M}$. (We see its implications in Chapter 1). Let A and B be two space-time vectors: $A = A^0 + \vec{A}$, $B = B^0 + \vec{B}$. The scalar product in space-time $A \cdot B$ is:

$$A \cdot B = \frac{1}{2}(A\widehat{B} + B\widehat{A}) = \frac{1}{2}(\widehat{A}B + \widehat{B}A). \tag{A.48}$$

We indeed have:

$$\begin{aligned} A\widehat{B} + B\widehat{A} &= (A^0 + \vec{A})(B^0 - \vec{B}) + (B^0 + \vec{B})(A^0 - \vec{A}) \\ &= A^0B^0 - A^0\vec{B} + B^0\vec{A} - \vec{A}\vec{B} + A^0B^0 + A^0\vec{B} - B^0\vec{A} - \vec{B}\vec{A} \\ &= 2(A^0B^0 - \vec{A}\cdot\vec{B}) = 2A \cdot B, & \text{(A.49)} \\ \widehat{A}B + \widehat{B}A &= \widehat{A\widehat{B} + B\widehat{A}} = \widehat{2A \cdot B} = 2A \cdot B. & \text{(A.50)} \end{aligned}$$

A.3.6 Laws of electromagnetism with Cl_3

The simplest framework to express the complicated laws of electromagnetism is also Cl_3: We call $A = A^\dagger$ a space-time vector "potential" and we calculate, with the Dirac operator $\nabla = \sigma^\mu \partial_\mu = \nabla^\dagger$, as well as $A = A^0 + \vec{A}$, the electromagnetic field $F = \vec{E} + i\vec{H}$ associated to this potential. It is purely a 2-vector in space-time, the sum of the electric field $\vec{E}$ and the magnetic field $i\vec{H}$, an axial vector. The derivation of the potential A gives:

$$\begin{aligned} \vec{E} + i\vec{H} = F = \nabla\widehat{A} &= (\partial_0 - \vec{\partial})(A^0 - \vec{A}) = \partial_0 A^0 - \partial_0\vec{A} - \vec{\partial}A^0 + \vec{\partial}\vec{A} \\ &= (\partial_0 A^0 + \vec{\partial}\cdot\vec{A}) + (-\partial_0\vec{A} - \vec{\partial}A^0) + i\vec{\partial}\times\vec{A}, \\ \vec{E} &= -\partial_0\vec{A} - \vec{\partial}A^0; \ \vec{H} = \vec{\partial}\times\vec{A}. & \text{(A.51)} \end{aligned}$$

Thus we have obtained a bivector F, sum of only a vector $\vec{E}$ and of a pseudovector $i\vec{H}$, if and only if the Lorentz gauge condition $0 = \partial_0 A^0 + \vec{\partial}\cdot\vec{A}$ is satisfied. We also have:

$$\begin{aligned} \overline{F} &= (\partial_0 A^0 + \vec{\partial}\cdot\vec{A}) - (-\partial_0\vec{A} - \vec{\partial}A^0) - i\vec{\partial}\times\vec{A}, \\ F - \overline{F} &= 2[(-\partial_0\vec{A} - \vec{\partial}A^0) + i\vec{\partial}\times\vec{A}] = 2(\vec{E} + i\vec{H}) = 2F. & \text{(A.52)} \end{aligned}$$

We thus let:

$$F := \vec{E} + i\vec{H} := \frac{1}{2}(\nabla\widehat{A} - \overline{\nabla\widehat{A}}) = \frac{1}{2}(\nabla\widehat{A} - A^\dagger\overline{\nabla}). \tag{A.53}$$

This implies:

$$\widehat{F} = -F^\dagger = -\vec{E} + i\vec{H} = \frac{1}{2}(\widehat{\nabla}A - \widehat{A}\nabla). \tag{A.54}$$

$$\nabla\widehat{F} = \frac{1}{2}(\nabla\widehat{\nabla}A - \nabla\widehat{A}\nabla) = \frac{1}{2}(\Box A - \nabla\widehat{A}\nabla), \tag{A.55}$$

$$\begin{aligned} \Box := \nabla\widehat{\nabla} &= (\partial_0 - \vec{\partial})(\partial_0 + \vec{\partial}) = \partial_0\partial_0 - \vec{\partial}\vec{\partial} \\ &= \partial_0\partial_0 - \partial_1\partial_1 - \partial_2\partial_2 - \partial_3\partial_3 = \widehat{\nabla}\nabla. \end{aligned} \tag{A.56}$$

Moreover, we obtain:

$$\nabla\widehat{F} = \nabla(\widehat{\nabla}A) = (\nabla\widehat{\nabla})A = \Box A = A\Box, \tag{A.57}$$

$$\begin{aligned} (\nabla\widehat{F})^\dagger &= \frac{1}{2}(\nabla\widehat{\nabla}A - \nabla\widehat{A}\nabla)^\dagger = \frac{1}{2}(\Box A - \nabla\widehat{A}\nabla)^\dagger \\ &= \frac{1}{2}(A\Box - \nabla\widehat{A}\nabla) = \frac{1}{2}(\Box A - \nabla\widehat{A}\nabla) = \nabla\widehat{F}. \end{aligned} \tag{A.58}$$

Therefore $\mathrm{j} = \nabla\widehat{F}$ is also a space-time covariant vector, called a "current". We also have:

$$\mathrm{j} = \mathrm{j}^0 - \vec{\mathrm{j}} = \nabla\widehat{F} = (\partial_0 - \vec{\partial})(-\vec{E} + i\vec{H}) \tag{A.59}$$

$$= \vec{\partial}\cdot\vec{E} + (-\partial_0\vec{E} + \vec{\partial}\times\vec{H}) + i(\partial_0\vec{H} + \vec{\partial}\times\vec{E}) - i\vec{\partial}\cdot\vec{H}. \tag{A.60}$$

Separating the scalar, vector, pseudovector and pseudoscalar part, we obtain Maxwell's equations:

$$\mathrm{j}_0 = \vec{\partial}\cdot\vec{E}, \tag{A.61}$$

$$\vec{\mathrm{j}} = \partial_0\vec{E} - \vec{\partial}\times\vec{H}, \tag{A.62}$$

$$0 = \partial_0\vec{H} + \vec{\partial}\times\vec{E}, \tag{A.63}$$

$$0 = \vec{\partial}\cdot\vec{H}. \tag{A.64}$$

We also have:

$$\begin{aligned} -\nabla\widehat{A}\nabla &= (\vec{E} + i\vec{H})(-\partial_0 + \vec{\partial}) \\ &= \vec{\partial}\cdot\vec{E} + (-\partial_0\vec{E} + \vec{\partial}\times\vec{H}) - i(\partial_0\vec{H} + \vec{\partial}\times\vec{E}) + i\vec{\partial}\cdot\vec{H} \\ &= \vec{\partial}\cdot\vec{E} - (\partial_0\vec{E} - \vec{\partial}\times\vec{H}) = \mathrm{j} = \nabla\widehat{F} = \Box A. \end{aligned} \tag{A.65}$$

Another form of electromagnetism exists when $A = A^\dagger$ (space-time vector) is replaced by $iB = -iB^\dagger$ (space-time pseudovector):

$$\begin{aligned}\vec{E}_m + i\vec{H}_m &= F_m = \nabla\widehat{iB} = -i\nabla\widehat{B} \\ &= -i[(\partial_0 - \vec{\partial})(B^0 - \vec{B})] = -i[\partial_0 B^0 - \partial_0\vec{B} - \vec{\partial}B^0 + \vec{\partial}\vec{B}] \\ &= -i(\partial_0 B^0 + \vec{\partial}\cdot\vec{B}) - i(-\partial_0\vec{B} - \vec{\partial}B^0) + \vec{\partial}\times\vec{B} \\ \vec{H}_m &= \partial_0\vec{B} + \vec{\partial}B^0;\ \vec{E}_m = \vec{\partial}\times\vec{B}. \end{aligned} \tag{A.66}$$

Thus we have also obtained a bivector F_m, the sum of a vector $\vec{E}_m$ and of a pseudovector $i\vec{H}_m$, if and only if the gauge condition $0 = \partial_0 B^0 + \vec{\partial}\cdot\vec{B}$ is satisfied. We also have:

$$\begin{aligned} -iB\widehat{\nabla} = \overline{F}_m &= -i(\partial_0 B^0 + \vec{\partial}\cdot\vec{B}) + i(-\partial_0\vec{B} - \vec{\partial}B^0) - \vec{\partial}\times\vec{B} \\ F_m - \overline{F}_m &= 2[i(\partial_0\vec{B} + \vec{\partial}B^0) + \vec{\partial}\times\vec{B}] = 2(\vec{E}_m + i\vec{H}_m) = 2F_m. \end{aligned} \tag{A.67}$$

We thus let:

$$F_m := \vec{E}_m + i\vec{H}_m := \frac{1}{2}(\nabla\widehat{iB} - \overline{\nabla\widehat{iB}}) = \frac{i}{2}(-\nabla\widehat{B} + B^\dagger\overline{\nabla}). \tag{A.68}$$

This implies:

$$\widehat{F}_m = -F_m^\dagger = -\vec{E}_m + i\vec{H}_m = \frac{i}{2}(\widehat{\nabla}B - \widehat{B}\nabla). \tag{A.69}$$

$$\nabla\widehat{F}_m = \frac{i}{2}(\nabla\widehat{\nabla}B - \nabla\widehat{B}\nabla) = \frac{i}{2}(\Box B - \nabla\widehat{B}\nabla). \tag{A.70}$$

Moreover, we obtain:

$$\nabla\widehat{F}_m = i\nabla(\widehat{\nabla}B) = i(\nabla\widehat{\nabla})B = i\Box B = iB\Box, \tag{A.71}$$

$$(\nabla\widehat{F}_m)^\dagger = (iB\Box)^\dagger = -i\Box B = -\nabla\widehat{F}_m. \tag{A.72}$$

Therefore $i\mathrm{k} = \nabla\widehat{F}_m$ is also a covariant space-time pseudovector, called the “magnetic current”. We also have (with unusual signs):

$$\begin{aligned} i\mathrm{k} = -i\mathrm{k}^0 + i\vec{\mathrm{k}} &= \nabla\widehat{F}_m = (\partial_0 - \vec{\partial})(-\vec{E}_m + i\vec{H}_m) \\ &= \vec{\partial}\cdot\vec{E}_m + (-\partial_0\vec{E}_m + \vec{\partial}\times\vec{H}_m) + i(\partial_0\vec{H}_m + \vec{\partial}\times\vec{E}_m) - i\vec{\partial}\cdot\vec{H}_m. \end{aligned} \tag{A.73}$$

Separating the scalar, vector, pseudovector and pseudoscalar part, we obtain Maxwell’s magnetic equations:

$$0 = \vec{\partial}\cdot\vec{E}_m, \tag{A.74}$$

$$0 = -\partial_0\vec{E}_m + \vec{\partial}\times\vec{H}_m, \tag{A.75}$$

$$\vec{\mathrm{k}} = \partial_0\vec{H}_m + \vec{\partial}\times\vec{E}_m, \tag{A.76}$$

$$\mathrm{k}_0 = \vec{\partial}\cdot\vec{H}_m. \tag{A.77}$$

We also have:

$$\begin{aligned}-i\nabla\widehat{B}\nabla &= (\vec{E}_m + i\vec{H}_m)(\partial_0 - \vec{\partial}) \qquad \text{(A.78)}\\ &= -\vec{\partial}\cdot\vec{E}_m + (\partial_0\vec{E}_m - \vec{\partial}\times\vec{H}_m) + i(\partial_0\vec{H}_m + \vec{\partial}\times\vec{E}_m) - i\vec{\nabla}\cdot\vec{H}_m\\ &= -i\vec{\partial}\cdot\vec{H}_m + i(\partial_0\vec{H}_m + \vec{\partial}\times\vec{E}_m) = -i\mathrm{k}_0 + i\vec{\mathrm{k}} = \nabla\widehat{F}_m = \Box iB.\end{aligned}$$

The full electromagnetism, with both electric charges and magnetic monopoles, has thus the simple rules:

$$F = \nabla\widehat{A + iB};\ \mathrm{j} + i\mathrm{k} = \nabla\widehat{F};\ \Box(A + iB) = \mathrm{j} + i\mathrm{k}. \qquad \text{(A.79)}$$

A.4 Tensor densities

We use the electron wave as:

$$\phi = \sqrt{2}(\xi \;\; \widehat{\eta}) = \sqrt{2}\begin{pmatrix}\xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^*\end{pmatrix}, \qquad \text{(A.80)}$$

which gives

$$\widehat{\phi} = \sqrt{2}(\eta \;\; \widehat{\xi}) = \sqrt{2}\begin{pmatrix}\eta_1 & -\xi_2^* \\ \eta_2 & \xi_1^*\end{pmatrix}, \qquad \text{(A.81)}$$

$$\phi^\dagger = \sqrt{2}\begin{pmatrix}\xi^\dagger \\ \widehat{\eta}^\dagger\end{pmatrix} \;;\; \overline{\phi} = \sqrt{2}\begin{pmatrix}\eta^\dagger \\ \widehat{\xi}^\dagger\end{pmatrix} = \sqrt{2}\begin{pmatrix}\eta_1^* & \eta_2^* \\ -\xi_2 & \xi_1\end{pmatrix}. \qquad \text{(A.82)}$$

A.4.1 Calculation of Ω_1 and Ω_2, and the determinant

We have with Dirac matrices:

$$\begin{aligned}\Omega_1 &= \overline{\psi}\psi = \begin{pmatrix}\eta^\dagger & \xi^\dagger\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \eta^\dagger\xi + \xi^\dagger\eta = \eta_1^*\xi_1 + \eta_2^*\xi_2 + \xi_1^*\eta_1 + \xi_2^*\eta_2,\\ \Omega_2 &= \overline{\psi}(-i\gamma_5)\psi = \begin{pmatrix}\eta^\dagger & \xi^\dagger\end{pmatrix}\begin{pmatrix}-iI & 0 \\ 0 & iI\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = -i\eta^\dagger\xi + i\xi^\dagger\eta, \qquad \text{(A.83)}\\ \Omega_1 + i\Omega_2 &= 2\eta^\dagger\xi;\ \Omega_1 - i\Omega_2 = 2\xi^\dagger\eta;\ \Omega_2 = i(-\eta_1^*\xi_1 - \eta_2^*\xi_2 + \xi_1^*\eta_1 + \xi_2^*\eta_2).\end{aligned}$$

And with the Pauli algebra we obtain:

$$\phi\overline{\phi} = \overline{\phi}\phi = \det(\phi) = 2(\xi_1\eta_1^* + \xi_2\eta_2^*) = 2\eta^\dagger\xi = \Omega_1 + i\Omega_2, \qquad \text{(A.84)}$$

$$\widehat{\phi}\phi^\dagger = \phi^\dagger\widehat{\phi} = \det(\phi)^* = 2(\eta_1\xi_1^* + \eta_2\xi_2^*) = 2\xi^\dagger\eta = \Omega_1 - i\Omega_2. \qquad \text{(A.85)}$$

We also obtain, for any $\phi \in Cl_3$:

$$\phi\overline{\phi} = \overline{\phi}\phi = \det(\phi), \qquad \text{(A.86)}$$

$$\phi[\det(\phi)^{-1}\overline{\phi}] = 1;\ \phi^{-1} = \det(\phi)^{-1}\overline{\phi}. \qquad \text{(A.87)}$$

The second reason of our interest in Cl_3 comes from the subset Cl_3^* of the invertible elements, which satisfy $\det(M) \neq 0$ and form a multiplicative Lie group. Moreover this Lie group has the Cl_3 algebra as Lie algebra. This multiplicative Lie group is the invariance group used throughout this book. Most of the progress brought about by Clifford algebra in quantum physics comes from the use of this multiplication, which was not computable in the Dirac theory.

A.4.2 Calculation of D_μ^ν

This calculation also gives the R_μ^ν of 1.1.2. It is enough to replace ϕ by M, and this means $\sqrt{2}\begin{pmatrix}\xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^*\end{pmatrix}$ by $\begin{pmatrix}a & b \\ c & d\end{pmatrix}$. We first calculate the components of the vector $\mathrm{D}_0 = \mathrm{J}$. With the Dirac matrices we have

$$\mathrm{D}_0^\mu = \mathrm{J}^\mu = \overline{\psi}\gamma^\mu\psi = \begin{pmatrix}\eta^\dagger & \xi^\dagger\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu \\ \widehat{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \eta^\dagger\sigma^\mu\eta + \xi^\dagger\widehat{\sigma}^\mu\xi. \quad \text{(A.88)}$$

We see here that the J current is the sum of the D_R and D_L currents:

$$\mathrm{D}_0^\mu = \mathrm{J}^\mu = \mathrm{D}_R^\mu + \mathrm{D}_L^\mu;\ \mathrm{D}_R^\mu = \xi^\dagger\widehat{\sigma}^\mu\xi;\ \mathrm{D}_L^\mu = \eta^\dagger\sigma^\mu\eta, \quad \text{(A.89)}$$

This comes from:

$$\begin{aligned}\mathrm{D}_R &:= \phi\frac{1+\sigma_3}{2}\phi^\dagger = 2\begin{pmatrix}\xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^*\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\xi_1^* & \xi_2^* \\ -\eta_2 & \eta_1\end{pmatrix} = 2\begin{pmatrix}\xi_1^*\xi_1 & \xi_2^*\xi_1 \\ \xi_1^*\xi_2 & \xi_2^*\xi_2\end{pmatrix} \\ &= \xi_1^*\xi_1(1+\sigma_3) + \xi_2^*\xi_2(1-\sigma_3) + \xi_2^*\xi_1(\sigma_1 + i\sigma_2) + \xi_1^*\xi_2(\sigma_1 - i\sigma_2) \\ &= \xi^\dagger\xi + (\xi^\dagger\sigma_3\xi\sigma_3 + (\xi^\dagger\sigma_1\xi)\sigma_1 + (\xi^\dagger\sigma_2\xi)\sigma_2 = (\xi^\dagger\widehat{\sigma}^\mu\xi)\sigma_\mu. \quad \text{(A.90)}\end{aligned}$$

Similarly we have:

$$\begin{aligned}\mathrm{D}_L &:= \phi\frac{1-\sigma_3}{2}\phi^\dagger = 2\begin{pmatrix}\xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^*\end{pmatrix}\begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\xi_1^* & \xi_2^* \\ -\eta_2 & \eta_1\end{pmatrix} = 2\begin{pmatrix}\eta_2^*\eta_2 & -\eta_2^*\eta_1 \\ -\eta_1^*\eta_2 & \eta_1^*\eta_1\end{pmatrix} \\ &= \eta_2^*\eta_2(1+\sigma_3) + \eta_1^*\eta_1(1-\sigma_3) - \eta_2^*\eta_1(\sigma_1 + i\sigma_2) - \eta_1^*\eta_2(\sigma_1 - i\sigma_2) \\ &= \eta^\dagger\eta + (\eta^\dagger\sigma^3\eta\sigma_3 + (\eta^\dagger\sigma^1\eta)\sigma_1 + (\eta^\dagger\sigma^2\eta)\sigma_2 = (\eta^\dagger\sigma^\mu\eta)\sigma_\mu. \quad \text{(A.91)}\end{aligned}$$

And we have:

$$\mathrm{D}_R + \mathrm{D}_L = \phi\Big(\frac{1+\sigma_3}{2} + \frac{1-\sigma_3}{2}\Big)\phi^\dagger = \phi\phi^\dagger = \mathrm{D}_0 = \mathrm{J}, \quad \text{(A.92)}$$

$$\mathrm{D}_R - \mathrm{D}_L = \phi\Big(\frac{1+\sigma_3}{2} - \frac{1-\sigma_3}{2}\Big)\phi^\dagger = \phi\sigma_3\phi^\dagger = \mathrm{D}_3 = \mathrm{K}. \quad \text{(A.93)}$$

This gives:

$$\mathrm{D}_0^0 = \mathrm{J}^0 = \xi^\dagger\sigma_0\xi + \eta^\dagger\sigma_0\eta = \xi_1\xi_1^* + \xi_2\xi_2^* + \eta_1\eta_1^* + \eta_2\eta_2^*, \quad \text{(A.94)}$$

$$\mathrm{D}_0^1 = \mathrm{J}^1 = \xi^\dagger\sigma_1\xi - \eta^\dagger\sigma_1\eta = \xi_1\xi_2^* + \xi_2\xi_1^* - \eta_1\eta_2^* - \eta_2\eta_1^*, \quad \text{(A.95)}$$

$$\mathrm{D}_0^2 = \mathrm{J}^2 = \xi^\dagger\sigma_2\xi - \eta^\dagger\sigma_2\eta = i(\xi_1\xi_2^* - \xi_2\xi_1^* - \eta_1\eta_2^* + \eta_2\eta_1^*), \quad \text{(A.96)}$$

$$\mathrm{D}_0^3 = \mathrm{J}^3 = \xi^\dagger\sigma_3\xi - \eta^\dagger\sigma_3\eta = \xi_1\xi_1^* - \xi_2\xi_2^* - \eta_1\eta_1^* + \eta_2\eta_2^*. \quad \text{(A.97)}$$

Now beginning with the tensors known in the formalism of Dirac matrices, we use (more details in B.1.1):

$$\gamma^0\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{A.98}$$

$$\gamma^j\gamma_5 = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \ j = 1, 2, 3, \tag{A.99}$$

$$\mathrm{K} = \mathrm{K}^\mu \sigma_\mu = (\xi^\dagger \widehat{\sigma}^\mu \xi)\sigma_\mu - (\eta^\dagger \sigma^\mu \eta)\sigma_\mu = \mathrm{D}_R - \mathrm{D}_L = \mathrm{D}_3. \tag{A.100}$$

We thus obtain:

$$\mathrm{D}_3^0 = \xi_1\xi_1^* + \xi_2\xi_2^* - \eta_1\eta_1^* - \eta_2\eta_2^*, \tag{A.101}$$

$$\mathrm{D}_3^3 = \xi_1\xi_1^* - \xi_2\xi_2^* + \eta_1\eta_1^* - \eta_2\eta_2^*, \tag{A.102}$$

$$\mathrm{D}_3^1 = \xi_1\xi_2^* + \xi_2\xi_1^* + \eta_1\eta_2^* + \eta_2\eta_1^*, \tag{A.103}$$

$$\mathrm{D}_3^2 = i(\xi_1\xi_2^* - \xi_2\xi_1^* + \eta_1\eta_2^* - \eta_2\eta_1^*). \tag{A.104}$$

For the calculation of components of D_1 and D_2, which are unknown in the formalism of Dirac matrices, we directly use the Pauli algebra:

$$\begin{aligned}
&\mathrm{D}_1 + i\mathrm{D}_2 = \phi(\sigma_1 + i\sigma_2)\phi^\dagger \\
&= 2\begin{pmatrix} \xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^* \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1^* & \xi_2^* \\ -\eta_2 & \eta_1 \end{pmatrix} = 4\begin{pmatrix} -\eta_2\xi_1 & \eta_1\xi_1 \\ -\eta_2\xi_2 & \eta_1\xi_2 \end{pmatrix} \qquad \text{(A.105)} \\
&= 2[-\eta_2\xi_1(1+\sigma_3) + \eta_1\xi_2(1-\sigma_3) + \eta_1\xi_1(\sigma_1 + i\sigma_2) - \eta_2\xi_2(\sigma_1 - i\sigma_2)] \\
&= 2[\widehat{\eta}^\dagger\xi + (\widehat{\eta}^\dagger\sigma_3\xi)\sigma_3 + (\widehat{\eta}^\dagger\sigma_1\xi)\sigma_1 + (\widehat{\eta}^\dagger\sigma_2\xi)\sigma_2 \\
&\mathrm{D}_1 + i\mathrm{D}_2 = 2(\widehat{\eta}^\dagger\widehat{\sigma}^\mu\xi)\sigma_\mu; \ \widehat{\eta}^\dagger = (-\eta_2 \ \ \eta_1). \qquad \text{(A.106)}
\end{aligned}$$

Similarly we obtain:

$$\begin{aligned}
&\mathrm{D}_1 - i\mathrm{D}_2 = \phi(\sigma_1 - i\sigma_2)\phi^\dagger \\
&= 2\begin{pmatrix} \xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^* \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} \xi_1^* & \xi_2^* \\ -\eta_2 & \eta_1 \end{pmatrix} = 4\begin{pmatrix} -\xi_1^*\eta_2^* & -\xi_2^*\eta_2^* \\ \xi_1^*\eta_1^* & \xi_2^*\eta_1^* \end{pmatrix} \qquad \text{(A.107)} \\
&= 2[-\xi_1^*\eta_2^*(1+\sigma_3) + \xi_2^*\eta_1^*(1-\sigma_3) - \xi_2^*\eta_2^*(\sigma_1 + i\sigma_2) + \xi_2^*\eta_1^*(\sigma_1 - i\sigma_2)] \\
&= 2[\xi^\dagger\widehat{\eta} + (\xi^\dagger\sigma_3\widehat{\eta})\sigma_3 + (\xi^\dagger\sigma_1\widehat{\eta})\sigma_1 + (\xi^\dagger\sigma_2\widehat{\eta})\sigma_2 \\
&\mathrm{D}_1 - i\mathrm{D}_2 = 2(\xi^\dagger\widehat{\sigma}^\mu\widehat{\eta})\sigma_\mu. \qquad \text{(A.108)}
\end{aligned}$$

Thus by adding and subtracting we get:

$$\mathrm{D}_1^0 = -\xi_1^*\eta_2^* - \xi_1\eta_2 + \xi_2^*\eta_1^* + \xi_2\eta_1, \tag{A.109}$$

$$\mathrm{D}_1^3 = -\xi_1^*\eta_2^* - \xi_1\eta_2 - \xi_2^*\eta_1^* - \xi_2\eta_1, \tag{A.110}$$

$$\mathrm{D}_1^1 = \xi_1^*\eta_1^* - \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1, \tag{A.111}$$

$$\mathrm{D}_1^2 = i(-\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1). \tag{A.112}$$

$$\mathrm{D}_2^0 = i(-\xi_1^*\eta_2^* + \xi_1\eta_2 + \xi_2^*\eta_1^* - \xi_2\eta_1), \tag{A.113}$$
$$\mathrm{D}_2^3 = i(-\xi_1^*\eta_2^* + \xi_1\eta_2 - \xi_2^*\eta_1^* + \xi_2\eta_1), \tag{A.114}$$
$$\mathrm{D}_2^1 = i(\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* - \xi_1\eta_1), \tag{A.115}$$
$$\mathrm{D}_2^2 = \xi_1^*\eta_1^* + \xi_2\eta_2 + \xi_2^*\eta_2^* + \xi_1\eta_1. \tag{A.116}$$

A.4.3 Calculation of S_k

For the calculation of $S = S_3$, the formalism of Dirac matrices gives with $S^{\mu\nu} = \overline{\psi} i\gamma^\mu\gamma^\nu\psi$:

$$E_3^3 := S_3^{12} = \overline{\psi} i\gamma^1\gamma^2\psi. \tag{A.117}$$

And we have:

$$i\gamma^1\gamma^2 = i\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \tag{A.118}$$

and similarly we have:

$$i\gamma^2\gamma^3 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \; ; \; i\gamma^3\gamma^1 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{A.119}$$

We then get:

$$E_3^3 := S_3^{12} = \begin{pmatrix} \eta^\dagger & \xi^\dagger \end{pmatrix}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \eta^\dagger\sigma_3\xi + \xi^\dagger\sigma_3\eta \tag{A.120}$$
$$= \eta_1^*\xi_1 - \eta_2^*\xi_2 + \xi_1^*\eta_1 - \xi_2^*\eta_2.$$

And similarly:

$$E_3^1 := S_3^{23} = \eta^\dagger\sigma_1\xi + \xi^\dagger\sigma_1\eta = \eta_1^*\xi_2 + \eta_2^*\xi_1 + \xi_1^*\eta_2 + \xi_2^*\eta_1, \tag{A.121}$$
$$E_3^2 := S_3^{31} = \eta^\dagger\sigma_2\xi + \xi^\dagger\sigma_2\eta = i(-\eta_1^*\xi_2 + \eta_2^*\xi_1 - \xi_1^*\eta_2 + \xi_2^*\eta_1). \tag{A.122}$$

Next we have:

$$\gamma^1\gamma^0 = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \tag{A.123}$$

which gives:

$$H_3^1 := S_3^{10} = \begin{pmatrix} \eta^\dagger & \xi^\dagger \end{pmatrix}\begin{pmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = -i\eta^\dagger\sigma_1\xi + i\xi^\dagger\sigma_1\eta \tag{A.124}$$
$$= i(-\eta_1^*\xi_2 - \eta_2^*\xi_1 + \xi_1^*\eta_2 + \xi_2^*\eta_1).$$

Similarly we have:

$$H_3^2 := S_3^{20} = -i\eta^\dagger\sigma_2\xi + i\xi^\dagger\sigma_2\eta = -\eta_1^*\xi_2 - \eta_2^*\xi_1 + \xi_1^*\eta_2 - \xi_2^*\eta_1, \tag{A.125}$$
$$H_3^3 := S_3^{30} = -i\eta^\dagger\sigma_3\xi + \xi^\dagger\sigma_3\eta = i(-\eta_1^*\xi_1 + \eta_2^*\xi_2 + \xi_1^*\eta_1 - \xi_2^*\eta_2). \tag{A.126}$$

We derive that:

$$S_3^{12} + iS_3^{30} = 2\eta^\dagger \sigma_3 \xi = 2(\xi_1 \eta_1^* - \xi_2 \eta_2^*), \tag{A.127}$$
$$S_3^{23} + iS_3^{10} = 2\eta^\dagger \sigma_1 \xi = 2(\xi_2 \eta_1^* - \xi_1 \eta_2^*), \tag{A.128}$$
$$S_3^{31} + iS_3^{20} = 2\eta^\dagger \sigma_2 \xi = 2i(-\xi_2 \eta_1^* + \xi_1 \eta_2^*), \tag{A.129}$$
$$S_3^{23} + iS_3^{10} + iS_3^{31} - S_3^{20} = 4\xi_2 \eta_1^*, \tag{A.130}$$
$$S_3^{23} + iS_3^{10} - iS_3^{31} + S_3^{20} = 4\xi_1 \eta_2^*. \tag{A.131}$$

And we have:

$$\begin{aligned}
&S_3^{23}\sigma_1 + S_3^{31}\sigma_2 + S_3^{12}\sigma_3 + S_3^{10} i\sigma_1 + S_3^{20} i\sigma_2 + S_3^{30} i\sigma_3 \\
&= \begin{pmatrix} S_3^{12} + iS_3^{30} & S_3^{23} + iS_3^{10} - iS_3^{31} + S_3^{20} \\ S_3^{23} + iS_3^{10} + iS_3^{31} - S_3^{20} & -(S_3^{12} + iS_3^{30}) \end{pmatrix} \\
&= 2 \begin{pmatrix} \xi_1\eta_1^* - \xi_2\eta_2^* & 2\xi_1\eta_2^* \\ 2\xi_2\eta_1^* & -(\xi_1\eta_1^* - \xi_2\eta_2^*) \end{pmatrix} \\
&= 2 \begin{pmatrix} \xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \eta_1^* & \eta_2^* \\ -\xi_2 & \xi_1 \end{pmatrix} \\
&= \phi \sigma_3 \overline{\phi} = S = S_3,
\end{aligned} \tag{A.132}$$

For the calculation of the components of S_1 and S_2, which are unknown in the formalism of Dirac matrices, we start directly from the Pauli algebra. We also need:

$$R := \phi \frac{1+\sigma_3}{2} = \sqrt{2} \begin{pmatrix} \xi_1 & 0 \\ \xi_2 & 0 \end{pmatrix}; \; L := \phi \frac{1-\sigma_3}{2} = \sqrt{2} \begin{pmatrix} 0 & -\eta_2^* \\ 0 & \eta_1^* \end{pmatrix},$$
$$S_R := \frac{1}{2}(S_1 + iS_2) = \phi \frac{1}{2}(\sigma_1 + i\sigma_2)\overline{\phi} = R\frac{1}{2}(\sigma_1 + i\sigma_2)\overline{R}, \tag{A.133}$$
$$S_L := \frac{1}{2}(S_1 - iS_2) = \phi \frac{1}{2}(\sigma_1 - i\sigma_2)\overline{\phi} = L\frac{1}{2}(\sigma_1 - i\sigma_2)\overline{L}. \tag{A.134}$$

We let:

$$S_R := \vec{E}_R + i\vec{H}_R; \; \vec{E}_R := E_R^j \sigma_j; \; \vec{H}_R := H_R^j \sigma_j, \tag{A.135}$$
$$E_R^1 := S_R^{23}; \; E_R^2 := S_R^{31}; \; E_R^3 := S_R^{12}; \; H_R^j := S_R^{j0}, \tag{A.136}$$
$$S_L := \vec{E}_L + i\vec{H}_L; \; \vec{E}_L := E_L^j \sigma_j; \; \vec{H}_L := H_L^j \sigma_j, \tag{A.137}$$
$$E_L^1 := S_L^{23}; \; E_L^2 := S_L^{31}; \; E_L^3 := S_L^{12}; \; H_L^j := S_L^{j0}. \tag{A.138}$$

This gives:

$$E_R^1 = \frac{1}{2}(\xi_1^2 - \xi_2^2 + \bar{\xi}_1^2 - \bar{\xi}_2^2);\ H_R^1 = \frac{i}{2}(-\xi_1^2 + \xi_2^2 + \bar{\xi}_1^2 - \bar{\xi}_2^2), \tag{A.139}$$
$$E_R^2 = \frac{i}{2}(\xi_1^2 + \xi_2^2 - \bar{\xi}_1^2 - \bar{\xi}_2^2);\ H_R^2 = \frac{1}{2}(\xi_1^2 + \xi_2^2 + \bar{\xi}_1^2 + \bar{\xi}_2^2), \tag{A.140}$$
$$E_R^3 = -\xi_1\xi_2 - \bar{\xi}_1\bar{\xi}_2;\ H_R^3 = i(\xi_1\xi_2 - \bar{\xi}_1\bar{\xi}_2), \tag{A.141}$$
$$E_L^1 = \frac{1}{2}(\eta_1^2 - \eta_2^2 + \bar{\eta}_1^2 - \bar{\eta}_2^2);\ H_L^1 = \frac{i}{2}(\eta_1^2 - \eta_2^2 - \bar{\eta}_1^2 + \bar{\eta}_2^2), \tag{A.142}$$
$$E_L^2 = \frac{i}{2}(\eta_1^2 + \eta_2^2 - \bar{\eta}_1^2 - \bar{\eta}_2^2);\ H_L^2 = \frac{1}{2}(-\eta_1^2 - \eta_2^2 - \bar{\eta}_1^2 - \bar{\eta}_2^2), \tag{A.143}$$
$$E_L^3 = -\eta_1\eta_2 - \bar{\eta}_1\bar{\eta}_2;\ H_L^3 = i(-\eta_1\eta_2 + \bar{\eta}_1\bar{\eta}_2). \tag{A.144}$$

The link with S_1 and S_2 is:

$$\begin{aligned} &S_1 = \vec{E}_1 + i\vec{H}_1 = S_R + S_L = \vec{E}_R + i\vec{H}_R + \vec{E}_L + i\vec{H}_L, \\ &\vec{E}_1 = \vec{E}_R + \vec{E}_L;\ \vec{H}_1 = \vec{H}_R + \vec{H}_L, \end{aligned} \tag{A.145}$$
$$\begin{aligned} &S_2 = \vec{E}_2 + i\vec{H}_2 = -iS_R + iS_L = -i(\vec{E}_R + i\vec{H}_R) + i(\vec{E}_L + i\vec{H}_L), \\ &\vec{E}_2 = \vec{H}_R - \vec{H}_L;\ \vec{H}_2 = \vec{E}_L - \vec{E}_R. \end{aligned} \tag{A.146}$$

We then obtain:

$$S_1^{12} = -\xi_1\xi_2 - \eta_1\eta_2 - \xi_1^*\xi_2^* - \eta_1^*\eta_2^*, \tag{A.147}$$
$$S_1^{30} = i(\xi_1\xi_2 - \eta_1\eta_2 - \xi_1^*\xi_2^* + \eta_1^*\eta_2^*), \tag{A.148}$$
$$S_1^{23} = \frac{1}{2}(\xi_1^2 - \xi_2^2 + \eta_1^2 - \eta_2^2 + \xi_1^{*2} - \xi_2^{*2} + \eta_1^{*2} - \eta_2^{*2}), \tag{A.149}$$
$$S_1^{10} = \frac{i}{2}(-\xi_1^2 + \xi_2^2 + \eta_1^2 - \eta_2^2 + \xi_1^{*2} - \xi_2^{*2} - \eta_1^{*2} + \eta_2^{*2}), \tag{A.150}$$
$$S_1^{20} = \frac{1}{2}(\xi_1^2 + \xi_2^2 - \eta_1^2 - \eta_2^2 + \xi_1^{*2} + \xi_2^{*2} - \eta_1^{*2} - \eta_2^{*2}), \tag{A.151}$$
$$S_1^{31} = \frac{i}{2}(\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2 - \xi_1^{*2} - \xi_2^{*2} - \eta_1^{*2} - \eta_2^{*2}). \tag{A.152}$$

Similarly we have:

$$S_2^{12} = i(\xi_1\xi_2 + \eta_1\eta_2 - \xi_1^*\xi_2^* - \eta_1^*\eta_2^*), \tag{A.153}$$
$$S_2^{30} = \xi_1\xi_2 - \eta_1\eta_2 + \xi_1^*\xi_2^* - \eta_1^*\eta_2^*, \tag{A.154}$$
$$S_2^{23} = \frac{i}{2}(-\xi_1^2 + \xi_2^2 - \eta_1^2 + \eta_2^2 + \xi_1^{*2} - \xi_2^{*2} + \eta_1^{*2} - \eta_2^{*2}), \tag{A.155}$$
$$S_2^{10} = \frac{1}{2}(-\xi_1^2 + \xi_2^2 + \eta_1^2 - \eta_2^2 - \xi_1^{*2} + \xi_2^{*2} + \eta_1^{*2} - \eta_2^{*2}), \tag{A.156}$$
$$S_2^{20} = \frac{i}{2}(-\xi_1^2 - \xi_2^2 + \eta_1^2 + \eta_2^2 + \xi_1^{*2} + \xi_2^{*2} - \eta_1^{*2} - \eta_2^{*2}), \tag{A.157}$$
$$S_2^{31} = \frac{1}{2}(\xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2 + \xi_1^{*2} + \xi_2^{*2} + \eta_1^{*2} + \eta_2^{*2}). \tag{A.158}$$

We also obtain the number of 36 densities by remarking that there are 8 squares and $28 = 8 \times 7/2$ pairs.

Calculation of $\overline{D}_\mu^\nu$

Let ϕ be an invertible element in Cl_3^*, with determinant $\rho e^{i\beta}$. Let D and $\overline{D}$ be the similitudes satisfying:

$$D : \mathrm{x} \mapsto \mathrm{x}' = D(\mathrm{x}) = \phi \mathrm{x} \phi^\dagger \ ; \ \ \overline{D} : \mathrm{x} \mapsto \mathrm{x}' = \overline{D}(\mathrm{x}) = \overline{\phi} \mathrm{x} \widehat{\phi}. \tag{A.159}$$

Let P such that:

$$\phi = \sqrt{\rho} e^{i\frac{\beta}{2}} P, \tag{A.160}$$

and let Lo and $\overline{L}o$ the similitudes such that:

$$Lo : \mathrm{x} \mapsto \mathrm{x}' = Lo(\mathrm{x}) = P\mathrm{x}P^\dagger \ ; \ \ \overline{L}o : \mathrm{x} \mapsto \mathrm{x}' = \overline{L}o(\mathrm{x}) = \overline{P}\mathrm{x}\widehat{P}. \tag{A.161}$$

We have:

$$\rho e^{i\beta} = \det(\phi) = \phi\overline{\phi} = \sqrt{\rho} e^{i\frac{\beta}{2}} P \sqrt{\rho} e^{i\frac{\beta}{2}} \overline{P} = \rho e^{i\beta} P\overline{P}, \tag{A.162}$$

then we get:

$$P\overline{P} = 1 \ ; \ \ \overline{P} = P^{-1} \ ; \ \ \overline{L}o = Lo^{-1}. \tag{A.163}$$

P is thus an element in $SL(2, \mathbb{C})$ and Lo is a Lorentz transformation. We know that, for such a transformation, if we denote by (Lo) the matrix of Lo in an orthonormal basis and g the signature-matrix:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A.164}$$

we have the following, M^t being the transposed matrix [8] of M:

$$(Lo)^{-1} = g(Lo)^t g \ ; \ \ (\overline{L}o)g = g(Lo)^t. \tag{A.165}$$

And we also have:

$$D(\mathrm{x}) = \phi \mathrm{x} \phi^\dagger = \sqrt{\rho} e^{i\frac{\beta}{2}} P \mathrm{x} \sqrt{\rho} e^{-i\frac{\beta}{2}} P^\dagger = \rho P \mathrm{x} P^\dagger = \rho Lo(\mathrm{x}), \tag{A.166}$$

then:

$$D = \rho Lo \ ; \ \ (D) = \rho(Lo). \tag{A.167}$$

8. The transposition exchanges the rows and columns: if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then: $M^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. We have, for any matrices A and B, $(AB)^t = B^t A^t$ and $\det(A^t) = \det(A)$.

Similarly we have:

$$\overline{D}(\mathrm{x}) = \overline{M}\mathrm{x}\widehat{M} = \sqrt{\rho}e^{i\frac{\beta}{2}}\overline{P}\mathrm{x}\sqrt{\rho}e^{-i\frac{\beta}{2}}\widehat{P} = \rho\overline{P}\mathrm{x}\widehat{P} = \rho\overline{L}o(\mathrm{x}), \tag{A.168}$$

$$\overline{D} = \rho\overline{L}o \ ; \quad (\overline{D}) = \rho(\overline{L}o). \tag{A.169}$$

Multiplying (A.165) by ρ, we get:

$$(\overline{D})g = g(D)^t \ ; \quad (\overline{D}) = g(D)^t g. \tag{A.170}$$

This gives, for $j = 1,\ 2,\ 3$ and $k = 1,\ 2,\ 3$:

$$\overline{D}_0^0 = D_0^0 \ ; \quad \overline{D}_0^j = -D_j^0 \ ; \quad \overline{D}_j^0 = -D_0^j \ ; \quad \overline{D}_j^k = D_k^j \tag{A.171}$$

The result is: rows, like columns, of the matrix (D_μ^ν), are orthogonal, because we have for D and $\overline{D}$:

$$D_\mu = \phi\sigma_\mu\phi^\dagger = D_\mu^\nu\sigma_\nu \ ; \quad \overline{D}_\mu = \overline{\phi}\sigma_\mu\widehat{\phi} = \overline{D}_\mu^\nu\sigma_\nu, \tag{A.172}$$

$$D_\mu \cdot D_\nu = \overline{D}_\mu \cdot \overline{D}_\nu = \delta_{\mu\nu}\rho^2. \tag{A.173}$$

A.4.4 Proof of $\nabla = \overline{M}\nabla'\widehat{M}$

Since ϕ has the same structure as M, we will use same notation:

$$M = \sqrt{2}(\xi \ \ \widehat{\eta}) = \sqrt{2}\begin{pmatrix}\xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^*\end{pmatrix}, \tag{A.174}$$

and this gives:

$$\widehat{M} = \sqrt{2}(\eta \ \ \widehat{\xi}) = \sqrt{2}\begin{pmatrix}\eta_1 & -\xi_2^* \\ \eta_2 & \xi_1^*\end{pmatrix}, \tag{A.175}$$

$$M^\dagger = \sqrt{2}\begin{pmatrix}\xi_1^* & \xi_2^* \\ -\eta_2 & \eta_1\end{pmatrix} \ ; \ \overline{M} = \sqrt{2}\begin{pmatrix}\eta_1^* & \eta_2^* \\ -\xi_2 & \xi_1\end{pmatrix}. \tag{A.176}$$

We get:

$$\overline{M}\nabla'\widehat{M} = 2\begin{pmatrix}\eta_1^* & \eta_2^* \\ -\xi_2 & \xi_1\end{pmatrix}\begin{pmatrix}\partial_0' - \partial_3' & -\partial_1' + i\partial_2' \\ -\partial_1' - i\partial_2' & \partial_0' + \partial_3'\end{pmatrix}\begin{pmatrix}\eta_1 & -\xi_2^* \\ \eta_2 & \xi_1^*\end{pmatrix} = \begin{pmatrix}A & B \\ C & D\end{pmatrix} \tag{A.177}$$

The R_μ^ν are obtained by (A.94) to (A.104) giving the D_μ^ν, we have:

$$\begin{aligned}
A &= 2[(\eta_1\eta_1^* + \eta_2\eta_2^*)\partial_0' + (-\eta_1\eta_2^* - \eta_2\eta_1^*)\partial_1' \\
&\quad + i(\eta_2\eta_1^* - \eta_1\eta_2^*)\partial_2' + (-\eta_1\eta_1^* + \eta_2\eta_2^*)\partial_3'] \\
&= (R_0^0 - R_3^0)\partial_0' + (R_0^1 - R_3^1)\partial_1' + (R_0^2 - R_3^2)\partial_2' + (R_0^3 - R_3^3)\partial_3' \\
&= R_0^\mu\partial_\mu' - R_3^\mu\partial_\mu' = \partial_0 - \partial_3. && \text{(A.178)} \\
C &= 2[(\xi_1\eta_2 - \xi_2\eta_1)\partial_0' + (-\xi_1\eta_1 + \xi_2\eta_2)\partial_1' \\
&\quad - i(\xi_1\eta_1 + \xi_2\eta_2)\partial_2' + (\xi_1\eta_2 + \xi_2\eta_1)\partial_3'] \\
&= (-R_1^0 - iR_2^0)\partial_0' + (-R_1^1 - iR_2^1)\partial_1' + (-R_1^2 - iR_2^2)\partial_2' + (-R_1^3 - iR_2^3)\partial_3' \\
&= -R_1^\mu\partial_\mu' - iR_2^\mu\partial_\mu' = -\partial_1 - i\partial_2. && \text{(A.179)} \\
B &= 2[(\xi_1^*\eta_2^* - \xi_2^*\eta_1^*)\partial_0' + (-\xi_1^*\eta_1^* + \xi_2^*\eta_2^*)\partial_1' \\
&\quad + i(\xi_1^*\eta_1^* + \xi_2^*\eta_2^*)\partial_2' + (\xi_1^*\eta_2^* + \xi_2^*\eta_1^*)\partial_3'] \\
&= (-R_1^0 + iR_2^0)\partial_0' + (-R_1^1 + iR_2^1)\partial_1' + (-R_1^2 + iR_2^2)\partial_2' + (-R_1^3 + iR_2^3)\partial_3' \\
&= -R_1^\mu\partial_\mu' + iR_2^\mu\partial_\mu' = -\partial_1 + i\partial_2. && \text{(A.180)} \\
D &= 2[(\xi_1\xi_1^* + \xi_2\xi_2^*)\partial_0' + (\xi_1\xi_2^* + \xi_2\xi_1^*)\partial_1' \\
&\quad + i(\xi_1\xi_2^* - \xi_2\xi_1^*)\partial_2' + (\xi_1\xi_1^* - \xi_2\xi_2^*)\partial_3'] \\
&= (R_0^0 + R_3^0)\partial_0' + (R_0^1 + R_3^1)\partial_1' + (R_0^2 + R_3^2)\partial_2' + (R_0^3 + R_3^3)\partial_3' \\
&= R_0^\mu\partial_\mu' + R_3^\mu\partial_\mu' = \partial_0 + \partial_3. && \text{(A.181)}
\end{aligned}$$

So we get:

$$\overline{M}\nabla'\widehat{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{pmatrix} = \nabla. \tag{A.182}$$

A.4.5 Proof of $\det(R_\mu^\nu) = r^4$

We let:

$$\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} := \begin{pmatrix} \mathrm{x}^0 + \mathrm{x}^3 & \mathrm{x}^1 - i\mathrm{x}^2 \\ \mathrm{x}^1 + i\mathrm{x}^2 & \mathrm{x}^0 - \mathrm{x}^3 \end{pmatrix};\ \begin{pmatrix} y_1' & y_2' \\ y_3' & y_4' \end{pmatrix} := \begin{pmatrix} \mathrm{x}^{0'} + \mathrm{x}^{3'} & \mathrm{x}^{1'} - i\mathrm{x}^{2'} \\ \mathrm{x}^{1'} + i\mathrm{x}^{2'} & \mathrm{x}^{0'} - \mathrm{x}^{3'} \end{pmatrix}, \tag{A.183}$$

$$\begin{aligned}
Y &:= \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{x}^0 \\ \mathrm{x}^1 \\ \mathrm{x}^2 \\ \mathrm{x}^3 \end{pmatrix} = NX, \\
Y' &:= \begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{x}^{0'} \\ \mathrm{x}^{1'} \\ \mathrm{x}^{2'} \\ \mathrm{x}^{3'} \end{pmatrix} = NX'.
\end{aligned} \tag{A.184}$$

We then have:

$$X = N^{-1}Y \ ; \quad X' = N^{-1}Y'. \tag{A.185}$$

We also let:

$$Y' = PY \ ; \quad X' = DX. \tag{A.186}$$

We get:

$$PNX = PY = Y' = NX' = NDX \ ; \quad PN = ND \ ; \quad D = N^{-1}PN, \tag{A.187}$$

which implies:

$$\det(R^\nu_\mu) = \det(N^{-1}PN) = \det(N^{-1})\det(P)\det(N) = \det(P). \tag{A.188}$$

We have:

$$\begin{pmatrix} y'_1 & y'_2 \\ y'_3 & y'_4 \end{pmatrix} = \mathrm{x}' = M\mathrm{x}M^\dagger = 2\begin{pmatrix} \xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^* \end{pmatrix}\begin{pmatrix} y_1 & y_2^* \\ y_3 & y_4 \end{pmatrix}\begin{pmatrix} \xi_1^* & \xi_2^* \\ -\eta_2 & \eta_1 \end{pmatrix} \tag{A.189}$$

$$= 2\begin{pmatrix} \begin{matrix} \xi_1\xi_1^* y_1 - \eta_2^*\xi_1^* y_3 \\ -\xi_1\eta_2 y_2 + \eta_2^*\eta_2 y_4 \end{matrix} & \begin{matrix} \xi_1\xi_2^* y_1 - \eta_2^*\xi_2^* y_3 \\ +\xi_1\eta_1 y_2 - \eta_2^*\eta_1 y_4 \end{matrix} \\ & \\ \begin{matrix} \xi_2\xi_1^* y_1 + \eta_1^*\xi_1^* y_3 \\ -\xi_2\eta_2 y_2 - \eta_1^*\eta_2 y_4 \end{matrix} & \begin{matrix} \xi_2\xi_2^* y_1 + \xi_2^*\eta_1^* y_3 \\ +\xi_2\eta_1 y_2 + \eta_1^*\eta_1 y_4 \end{matrix} \end{pmatrix},$$

which gives:

$$Y' = PY \ ; \quad P = 2\begin{pmatrix} \xi_1\xi_1^* & -\xi_1\eta_2 & -\xi_1^*\eta_2^* & \eta_2\eta_2^* \\ \xi_1\xi_2^* & \xi_1\eta_1 & -\xi_2^*\eta_2^* & -\eta_1\eta_2^* \\ \xi_2\xi_1^* & -\xi_2\eta_2 & \xi_1^*\eta_1^* & -\eta_2\eta_1^* \\ \xi_2\xi_2^* & \xi_2\eta_1 & \xi_2^*\eta_1^* & \eta_1\eta_1^* \end{pmatrix}. \tag{A.190}$$

The calculation of the determinant of P thus gives:

$$\begin{aligned} \det(P) &= 16(\xi_1^2\xi_1^{*2}\eta_1^2\eta_1^{*2} + \xi_1^2\xi_2^{*2}\eta_2^2\eta_1^{*2} + \xi_2^2\xi_1^{*2}\eta_1^2\eta_2^{*2} + \xi_2^2\xi_2^{*2}\eta_2^2\eta_2^{*2} \\ &\quad + 2\xi_1^2\xi_1^*\xi_2^*\eta_1\eta_1^{*2}\eta_2 + 4\xi_1\xi_1^*\xi_2\xi_2^*\eta_1\eta_1^*\eta_2\eta_2^* \\ &\quad + 2\xi_1\xi_1^{*2}\xi_2\eta_1^2\eta_1^*\eta_2^* + 2\xi_1\xi_2\xi_2^{*2}\eta_1^*\eta_2^2\eta_2^* + 2\xi_1^*\xi_2^2\xi_2^*\eta_1\eta_2\eta_2^{*2}) \\ &= 16(\xi_1\xi_1^*\eta_1\eta_1^* + \xi_1\xi_2^*\eta_1^*\eta_2 + \xi_1^*\xi_2\eta_1\eta_2^* + \xi_2\xi_2^*\eta_2\eta_2^*)^2 \\ &= 16[(\xi_1\eta_1^* + \xi_2\eta_2^*)(\xi_1^*\eta_1 + \xi_2^*\eta_2)]^2. \end{aligned} \tag{A.191}$$

Thus we get:

$$\begin{aligned} \det(R^\nu_\mu) &= [2(\xi_1\eta_1^* + \xi_2\eta_2^*)2(\xi_1^*\eta_1 + \xi_2^*\eta_2)]^2 \\ &= [re^{i\theta}re^{-i\theta}]^2 = (r^2)^2 = r^4. \end{aligned} \tag{A.192}$$

A.4.6 Relations between tensors

We have:

$$\mathrm{D}_\mu\widehat{\mathrm{D}}_\nu = \phi\sigma_\mu\phi^\dagger\widehat{\phi\sigma_\nu\phi^\dagger} = \phi\sigma_\mu\phi^\dagger\widehat{\phi}\widehat{\sigma}_\nu\widehat{\phi}^\dagger \tag{A.193}$$
$$= \phi\sigma_\mu(\Omega_1 - i\Omega_2)\widehat{\sigma}_\nu\overline{\phi} = (\Omega_1 - i\Omega_2)\phi\sigma_\mu\widehat{\sigma}_\nu\overline{\phi}. \tag{A.194}$$

For $j = 1,\ 2,\ 3$ this gives:

$$\mathrm{D}_0\widehat{\mathrm{D}}_j = (\Omega_1 - i\Omega_2)\phi\widehat{\sigma}_j\overline{\phi} = -(\Omega_1 - i\Omega_2)S_j, \tag{A.195}$$
$$\mathrm{D}_j\widehat{\mathrm{D}}_0 = (\Omega_1 - i\Omega_2)\phi\sigma_j\overline{\phi} = (\Omega_1 - i\Omega_2)S_j, \tag{A.196}$$
$$\mathrm{D}_1\widehat{\mathrm{D}}_2 = (\Omega_1 - i\Omega_2)\phi\sigma_1\widehat{\sigma}_2\overline{\phi} = (\Omega_1 - i\Omega_2)\phi(-i)\sigma_3\overline{\phi} = -(\Omega_2 + i\Omega_1)S_3, \tag{A.197}$$
$$\mathrm{D}_2\widehat{\mathrm{D}}_1 = (\Omega_1 - i\Omega_2)\phi\sigma_2\widehat{\sigma}_1\overline{\phi} = (\Omega_1 - i\Omega_2)\phi i\sigma_3\overline{\phi} = (\Omega_2 + i\Omega_1)S_3. \tag{A.198}$$

And similarly we get:

$$\mathrm{D}_2\widehat{\mathrm{D}}_3 = -\mathrm{D}_3\widehat{\mathrm{D}}_2 = -(\Omega_2 + i\Omega_1)S_1, \tag{A.199}$$
$$\mathrm{D}_3\widehat{\mathrm{D}}_1 = -\mathrm{D}_1\widehat{\mathrm{D}}_3 = -(\Omega_2 + i\Omega_1)S_2. \tag{A.200}$$

For $j = 1,\ 2,\ 3$ and for $k = 1,\ 2,\ 3$, we have:

$$\mathrm{D}_j\widehat{S}_k = \phi\sigma_j\phi^\dagger\widehat{\phi\sigma_k\overline{\phi}} = \phi\sigma_j\phi^\dagger\widehat{\phi}\widehat{\sigma}_k\phi^\dagger = -(\Omega_1 - i\Omega_2)\phi\sigma_j\sigma_k\phi^\dagger, \tag{A.201}$$
$$S_j\mathrm{D}_k = \phi\sigma_j\overline{\phi}\phi\sigma_k\phi^\dagger = (\Omega_1 + i\Omega_2)\phi\sigma_j\sigma_k\phi^\dagger. \tag{A.202}$$

Thus for $j = 1,\ 2,\ 3$ we get:

$$\mathrm{D}_j\widehat{S}_j = -(\Omega_1 - i\Omega_2)\phi\phi^\dagger = (-\Omega_1 + i\Omega_2)\mathrm{D}_0, \tag{A.203}$$
$$S_j\mathrm{D}_j = (\Omega_1 + i\Omega_2)\phi\phi^\dagger = (\Omega_1 + i\Omega_2)\mathrm{D}_0. \tag{A.204}$$

And for $k \neq j$ we have:

$$\mathrm{D}_1\widehat{S}_2 = -i(\Omega_1 - i\Omega_2)\phi\sigma_3\phi^\dagger = -(\Omega_2 + i\Omega_1)\mathrm{D}_3 = -\mathrm{D}_2\widehat{S}_1, \tag{A.205}$$
$$S_1\mathrm{D}_2 = i(\Omega_1 + i\Omega_2)\phi\sigma_3\phi^\dagger = (-\Omega_2 + i\Omega_1)\mathrm{D}_3, = -S_2\mathrm{D}_1, \tag{A.206}$$

$$\mathrm{D}_2\widehat{S}_3 = -i(\Omega_1 - i\Omega_2)\phi\sigma_1\phi^\dagger = -(\Omega_2 + i\Omega_1)\mathrm{D}_1 = -\mathrm{D}_3\widehat{S}_2, \tag{A.207}$$
$$S_2\mathrm{D}_3 = i(\Omega_1 + i\Omega_2)\phi\sigma_1\phi^\dagger = (-\Omega_2 + i\Omega_1)\mathrm{D}_1, = -S_3\mathrm{D}_2, \tag{A.208}$$

$$\mathrm{D}_3\widehat{S}_1 = -i(\Omega_1 - i\Omega_2)\phi\sigma_2\phi^\dagger = -(\Omega_2 + i\Omega_1)\mathrm{D}_2 = -\mathrm{D}_1\widehat{S}_3, \tag{A.209}$$
$$S_3\mathrm{D}_1 = i(\Omega_1 + i\Omega_2)\phi\sigma_2\phi^\dagger = (-\Omega_2 + i\Omega_1)\mathrm{D}_2 = -S_2\mathrm{D}_1. \tag{A.210}$$

For $j = 1,\ 2,\ 3$, we also have:

$$\begin{aligned} \mathrm{D}_0\widehat{S}_j &= \phi\phi^\dagger\widehat{\phi\sigma_j\overline{\phi}} = \phi\phi^\dagger\widehat{\overline{\phi}}\widehat{\sigma}_j\phi^\dagger = (-\Omega_1 + i\Omega_2)\phi\sigma_j\phi^\dagger \\ &= (-\Omega_1 + i\Omega_2)\mathrm{D}_j, \end{aligned} \tag{A.211}$$

$$S_j\mathrm{D}_0 = \phi\sigma_j\overline{\phi}\phi\phi^\dagger = (\Omega_1 + i\Omega_2)\phi\sigma_j\phi^\dagger = (\Omega_1 + i\Omega_2)\mathrm{D}_j. \tag{A.212}$$

Finally we have for $j = 1,\ 2,\ 3$ and for $k = 1,\ 2,\ 3$:

$$S_jS_k = \phi\sigma_j\overline{\phi}\phi\sigma_k\overline{\phi} = (\Omega_1 + i\Omega_2)\phi\sigma_j\sigma_k\overline{\phi}, \tag{A.213}$$

$$S_jS_j = (\Omega_1 + i\Omega_2)\phi\overline{\phi} = (\Omega_1 + i\Omega_2)^2. \tag{A.214}$$

While for $k \neq j$, we get:

$$S_1S_2 = -S_2S_1 = (-\Omega_2 + i\Omega_1)S_3, \tag{A.215}$$

$$S_2S_3 = -S_3S_2 = (-\Omega_2 + i\Omega_1)S_1, \tag{A.216}$$

$$S_3S_1 = -S_1S_3 = (-\Omega_2 + i\Omega_1)S_2. \tag{A.217}$$

Appendix B

Other Clifford algebras

We present the space-time algebra and the Dirac matrices. We study its link with the Pauli algebra and the link between the invariant wave equation and the Lagrangian density. We study the same with space-time algebra. We calculate Tétrode's tensor. Then we present the Clifford algebra $Cl_{3,3} = \mathrm{End}(Cl_3)$ that we need for the study of weak and strong interactions and gravitation.

B.1 Clifford algebra of space-time

The Clifford algebra of space-time $Cl_{1,3}$ contains the real numbers and the vectors of space-time $\mathbf{x}$ such that:

$$\mathbf{x} = \mathrm{x}^0\gamma_0 + \mathrm{x}^1\gamma_1 + \mathrm{x}^2\gamma_2 + \mathrm{x}^3\gamma_3 = \mathrm{x}^\mu\gamma_\mu. \tag{B.1}$$

The four γ_μ form an orthonormal basis of space-time:

$$(\gamma_0)^2 = 1 \;\; ; \;\; (\gamma_1)^2 = (\gamma_2)^2 = (\gamma_3)^2 = -1 \;\; ; \;\; \gamma_\mu \cdot \gamma_\nu = 0 \; , \; \mu \neq \nu. \tag{B.2}$$

Proponents of Clifford algebra can generally be divided into two camps: those who put a + sign for time (Hestenes [73][78]), and those who put a − sign for time (Deheuvels [60]). We will see in B.2 that these two signatures give two subalgebras of $Cl_{3,3}$. Here we use a + sign for time, which corresponds to the choice of Hestenes. It is necessary because the metric of space-time is given by the determinant (A.86). The general term of $Cl_{1,3}$ is a sum:

$$N = s + v + B + p_v + p_s \tag{B.3}$$

where s is a real number, v is a vector in space-time, B is a 2-vector, p_v is a 3-vector (or pseudovector) and p_s is a pseudoscalar. There are 1+4+6+4+1 = 16 = 2^4 dimensions on the real field because: There are 6 independent 2-vectors $\gamma_{01} = \gamma_0\gamma_1$, γ_{02}, γ_{03}, γ_{12}, γ_{23} and γ_{31}, where $\gamma_{ji} = -\gamma_{ij}$, $j \neq i$, and

4 3-vectors γ_{012}, γ_{023}, γ_{031} and γ_{123} and one pseudoscalar:

$$p_s = b\gamma_{0123} \ ; \ \gamma_{0123} = \gamma_0\gamma_1\gamma_2\gamma_3 = \mathbf{i} = i\gamma_5, \tag{B.4}$$

where b is a real number.

The even part of N is $s + B + p_s$, while the odd part is $v + p_v$. The main automorphism satisfies $N \mapsto \widehat{N} = s - v + B - p_v + p_s$. The reversion satisfies:

$$N \mapsto \widetilde{N} = s + v - B - p_v + p_s. \tag{B.5}$$

Among the 16 generators of $Cl_{1,3}$, $10 = 5 \times 4/2$ have a square of -1 and $6 = 4 \times 3/2$ have a square of 1:

$$\begin{aligned} 1^2 = {\gamma_{01}}^2 = {\gamma_{02}}^2 = {\gamma_{03}}^2 = {\gamma_0}^2 = {\gamma_{123}}^2 = 1, \\ {\gamma_1}^2 = {\gamma_2}^2 = {\gamma_3}^2 = {\gamma_{12}}^2 = {\gamma_{23}}^2 = {\gamma_{31}}^2 \\ = {\gamma_{012}}^2 = {\gamma_{023}}^2 = {\gamma_{031}}^2 = {\gamma_{0123}}^2 = -1. \end{aligned} \tag{B.6}$$

Two remarks:

1 - If we use a $+$ sign for space then we get 6 generators whose square is 1 and 10 whose square is -1. The two Clifford algebras $Cl_{1,3}$ and $Cl_{3,1}$ are hence not equal.

2 - The even subalgebra $Cl^+_{1,3}$, formed by all even elements $N = s+b+ps$ is 8-dimensional and is isomorphic to Cl_3. We will see this in detail in the next section using the Dirac matrices. The even subalgebra of $Cl_{3,1}$ is also isomorphic to Cl_3.

The privileged differential operator in $Cl_{1,3}$ is:

$$\boldsymbol{\partial} = \gamma^\mu \partial_\mu \ ; \ \gamma^0 = \gamma_0 \ ; \ \gamma^j = -\gamma_j \ , \ j = 1,\ 2,\ 3. \tag{B.7}$$

It satisfies:

$$\boldsymbol{\partial\partial} = \Box = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2. \tag{B.8}$$

B.1.1 Dirac matrices, electromagnetism

Most physicists do not directly use the Clifford algebra of space-time, but use instead the matrix algebra $M_4(\mathbb{C})$, an algebra on the complex field. This algebra is 16-dimensional on the complex field, and thus 32-dimensional on the real field. Therefore $M_4(\mathbb{C}) \neq Cl_{1,3}$. The Dirac matrices are not uniquely defined. The easiest way to link $Cl_{1,3}$ to Cl_3 makes use of[1] (1.4), which we recall here:

$$\gamma_0 = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix};\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ ;\ \gamma^j = -\gamma_j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix},\ j = 1,2,3.$$

1. This choice of Dirac matrices is not the same one used in Dirac theory to calculate the solutions for the hydrogen atom. But it is the choice made for high velocities and when special relativity is required. It is also the choice of electroweak theory. We will see in Appendix C that this choice also allows us to solve the equation in the case of the H atom by separation of variables in spherical coordinates. This also proves that the early choice of Dirac matrices was not necessary, only sufficient, for the H atom.

We then have:

$$\boldsymbol{\partial} = \gamma^\mu \partial_\mu = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix} \tag{B.9}$$

It is easy to show that:

$$\gamma_{0j} = \begin{pmatrix} -\sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} ; \ \gamma_{23} = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}$$
$$\mathbf{i} = \gamma_{0123} = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}. \tag{B.10}$$

Isomorphism between $Cl_{1,3}^+$ and Cl_3 : let N be any even element. With:

$$N = a + B + p_s \ \ ; \ \ B = u_1\gamma_{10} + u_2\gamma_{20} + u_3\gamma_{30} + v_1\gamma_{32} + v_2\gamma_{13} + v_3\gamma_{21},$$
$$p_s = b\gamma_{0123} = b\mathbf{i}, \tag{B.11}$$
$$M = a + \vec{u} + i\vec{v} + ib \ ; \ \ \vec{u} = u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3,$$
$$\vec{v} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3. \tag{B.12}$$

B is a bivector and p_s is a pseudoscalar in space-time. With the choice of (1.4), for the Dirac matrices we have:

$$N = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} \ ; \ \widetilde{N} = \begin{pmatrix} \overline{M} & 0 \\ 0 & M^\dagger \end{pmatrix}. \tag{B.13}$$

Since the $P : M \mapsto \widehat{M}$ conjugation is compatible with the addition and the multiplication, the algebra of M is isomorphic to the algebra of N. Since N contains both M and $\widehat{M}$, the Dirac matrices use the two nonequivalent representations of Cl_3^* (this is well known in Lie group theory [1]).

The Dirac operator is:

$$\boldsymbol{\partial} = \gamma^\mu \partial_\mu = \begin{pmatrix} 0 & \partial_0 - \vec{\partial} \\ \partial_0 + \vec{\partial} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix}. \tag{B.14}$$

Similarly, the electromagnetic potential is the vector:

$$\mathbf{A} := \gamma_\mu A^\mu = \begin{pmatrix} 0 & A^0 + \vec{A} \\ A^0 - \vec{A} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix}. \tag{B.15}$$

The electromagnetic field is the bivector:

$$\mathbf{F} := \boldsymbol{\partial} \wedge \mathbf{A} = \boldsymbol{\partial}\mathbf{A} - \mathbf{A}\boldsymbol{\partial} \tag{B.16}$$
$$= \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix} - \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix} \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \nabla\widehat{A} - A\widehat{\nabla} & 0 \\ 0 & \widehat{\nabla}A - \widehat{A}\nabla \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & \widehat{F} \end{pmatrix}.$$

The electric current satisfies:

$$\mathbf{j} = \boldsymbol{\partial}\mathbf{F} = \boldsymbol{\partial}\boldsymbol{\partial}\mathbf{A} = \Box\mathbf{A} = \begin{pmatrix} 0 & \nabla\widehat{F} \\ \widehat{\nabla}F & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{j} \\ \widehat{\mathrm{j}} & 0 \end{pmatrix}. \tag{B.17}$$

B.1.2 $Cl_{1,3}$ as Cartesian product $Cl_3 \times Cl_3$

We can shorten the calculations in $Cl_{1,3}$ by considering only the first row of the Dirac matrices. This is possible because the second row is obtained from the first one by using the P automorphism on Cl_3. The general element of $Cl_{1,3}$ may thus be expressed as a couple of elements of Cl_3:

$$M = (A \quad B);\ N = (C \quad D);\ M + N = (A + C \quad B + D), \tag{B.18}$$

$$MN = (AC + B\widehat{D} \quad AD + B\widehat{C}) \tag{B.19}$$

$$\gamma^\mu = (0 \quad \sigma^\mu);\ \mathbf{x} = \mathrm{x}^\mu \gamma_\mu = (0 \quad \mathrm{x}) = (0 \quad \mathrm{x}^\mu \sigma_\mu), \tag{B.20}$$

$$\boldsymbol{\partial} = \gamma^\mu \partial_\mu = (0 \quad \nabla) = (0 \quad \sigma^\mu \partial_\mu);\ \boldsymbol{\partial}(A \quad B) = (\nabla\widehat{B} \quad \nabla\widehat{A}). \tag{B.21}$$

With these notations, and for any $\mathbf{u}$ and $\mathbf{v}$ in space-time, we have:

$$\mathbf{uv} + \mathbf{vu} = (0 \quad \mathrm{u})(0 \quad \mathrm{v}) + (0 \quad \mathrm{v})(0 \quad \mathrm{u}) = (\mathrm{u}\widehat{\mathrm{v}} + \mathrm{v}\widehat{\mathrm{u}} \quad 0). \tag{B.22}$$

Identifying A and $(A \quad 0)$, we then have:

$$\mathbf{uv} + \mathbf{vu} = \mathrm{u}\widehat{\mathrm{v}} + \mathrm{v}\widehat{\mathrm{u}} = \mathrm{u}^0\mathrm{v}^0 - \mathrm{u}^1\mathrm{v}^1 - \mathrm{u}^2\mathrm{v}^2 - \mathrm{u}^3\mathrm{v}^3. \tag{B.23}$$

This identification allows us to consider $Cl_{1,3}$ as a Cl_3-module:

$$X(A \quad B) = (X \quad 0)(A \quad B) = (XA \quad XB), \tag{B.24}$$

for any X, A and B in Cl_3. This is what allows us the use in $Cl_{1,3}$ of the complex field, which is the center of Cl_3, nevertheless the fact that the center of $Cl_{1,3}$ is only the real field.

B.1.3 Proof of $R^\mu_\nu \gamma^\nu = \widetilde{N}\gamma^\mu N$

Using the aforementioned notation we have:

$$N = \begin{pmatrix} M & 0 \end{pmatrix}\ ;\ \ \widetilde{N} = \begin{pmatrix} \overline{M} & 0 \end{pmatrix}\ ;\ \ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \end{pmatrix}, \tag{B.25}$$

the equality $R^\mu_\nu \gamma^\nu = \widetilde{N}\gamma^\mu N$ is equivalent to:

$$\begin{pmatrix} 0 & R^\mu_\nu \sigma^\nu \end{pmatrix} = \begin{pmatrix} 0 & \overline{M}\sigma^\mu \widehat{M} \end{pmatrix}. \tag{B.26}$$

And the equality proved in A.4.4 : $\nabla = \overline{M}\nabla'\widehat{M}$, can also be expressed as:

$$\partial_\nu \sigma^\nu = \overline{M}\partial'_\mu \sigma^\mu \widehat{M}, \tag{B.27}$$

which means

$$R^\mu_\nu \partial'_\mu \sigma^\nu = \overline{M}\sigma^\mu \widehat{M}\partial'_\mu. \tag{B.28}$$

And we thus have:

$$R^\mu_\nu \sigma^\nu = \overline{M}\sigma^\mu \widehat{M};\ R^\mu_\nu \gamma^\nu = \widetilde{N}\gamma^\mu N. \tag{B.29}$$

B.1.4 Invariant equation and Lagrangian density

We will now prove that the Lagrangian density of the Dirac theory is the real part, in the sense of Clifford algebra, of the wave equation in its invariant form (1.113). Then noting $\langle M\rangle_n$ the n-vector part of M, we must prove that:

$$L = \langle \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\overline{\phi}\phi\rangle_0. \tag{B.30}$$

But we have:

$$\begin{aligned}\overline{\phi}A\widehat{\phi} &= A^\mu\overline{\phi}\sigma_\mu\widehat{\phi} = A_0\overline{D}_0 - \sum_{j=1}^{j=3}A_j\overline{D}_j = A_0(\overline{D}_0^\mu\sigma_\mu) - \sum_{j=1}^{j=3}A_j(\overline{D}_j^\mu\sigma_\mu)\\ &= A_0(\overline{D}_0^0 + \sum_{j=1}^{j=3}\overline{D}_0^j\sigma_j) - \sum_{j=1}^{j=3}A_j(\overline{D}_j^0 + \sum_{k=1}^{k=3}\overline{D}_j^k\sigma_k).\end{aligned} \tag{B.31}$$

We established with the calculation of the similitude $\overline{D}$ (A.171) that gives:

$$\overline{\phi}A\widehat{\phi} = A_0(D_0^0 - \sum_{j=1}^{j=3}D_j^0\sigma_j) - \sum_{j=1}^{j=3}A_j(-D_0^j + \sum_{k=1}^{k=3}D_k^j\sigma_k) = A_\nu D_\mu^\nu\sigma^\mu. \tag{B.32}$$

The scalar part is then

$$\langle\overline{\phi}A\widehat{\phi}\rangle_0 = D_0^\nu A_\nu = A_\nu \mathrm{J}^\nu = A_\mu\overline{\psi}\gamma^\mu\psi. \tag{B.33}$$

We next have $\overline{\phi}\phi = \Omega_1 + i\Omega_2$, then:

$$\langle m\overline{\phi}\phi\rangle_0 = m\Omega_1 = m\overline{\psi}\psi. \tag{B.34}$$

We next get:

$$\begin{aligned}&\frac{1}{2}[(\overline{\psi}\gamma^\mu(-i)\partial_\mu\psi) + (\overline{\psi}\gamma^\mu(-i)\partial_\mu\psi)^\dagger] = \frac{i}{2}(-\overline{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu\overline{\psi}\gamma^\mu\psi)\\ &= -\frac{i}{2}[\xi^\dagger(\widehat{\nabla}\xi) - (\xi^\dagger\widehat{\nabla})\xi + \eta^\dagger(\nabla\eta) - (\eta^\dagger\nabla)\eta].\end{aligned} \tag{B.35}$$

With Cl_3 we have:

$$\begin{aligned}&\frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) - (\overline{\phi}\nabla)\widehat{\phi}]\sigma_{21} = -\frac{i}{2}[\overline{\phi}(\nabla\widehat{\phi}\sigma_3) - (\overline{\phi}\nabla)\widehat{\phi}\sigma_3]\\ &= -i\begin{pmatrix}\eta^\dagger(\nabla\eta) - (\eta^\dagger\nabla)\eta & -\eta^\dagger(\nabla\widehat{\xi}) + (\eta^\dagger\nabla)\widehat{\xi}\\ \widehat{\xi}^\dagger(\nabla\eta) - (\widehat{\xi}^\dagger\nabla)\eta & -\widehat{\xi}^\dagger(\nabla\widehat{\xi}) + (\widehat{\xi}^\dagger\nabla)\widehat{\xi}\end{pmatrix},\\ &\langle\frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) - (\overline{\phi}\nabla)\widehat{\phi}]\sigma_{21}\rangle_0 = -\frac{i}{2}[\eta^\dagger(\nabla\eta) - (\eta^\dagger\nabla)\eta - \widehat{\xi}^\dagger(\nabla\widehat{\xi}) + (\widehat{\xi}^\dagger\nabla)\widehat{\xi}].\end{aligned} \tag{B.36}$$

And we have:

$$
\begin{aligned}
-\widehat{\xi}^{\dagger}(\nabla\widehat{\xi}) + (\widehat{\xi}^{\dagger}\nabla)\widehat{\xi} &= \overline{(\widehat{\xi}^{\dagger}\nabla)\widehat{\xi}} - \overline{\widehat{\xi}^{\dagger}(\nabla\widehat{\xi})} \\
&= \xi^{\dagger}(\widehat{\nabla}\xi) - (\xi^{\dagger}\widehat{\nabla})\xi.
\end{aligned} \tag{B.37}
$$

We thus have:

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}[(\overline{\psi}\gamma^{\mu}(-i)\partial_{\mu}\psi) + (\overline{\psi}\gamma^{\mu}(-i)\partial_{\mu}\psi)^{\dagger}] + qA_{\mu}\overline{\psi}\gamma^{\mu}\psi + m\overline{\psi}\psi \\
&= \langle\overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\overline{\phi}\phi\rangle_0.
\end{aligned} \tag{B.38}
$$

B.1.5 Calculation of Tétrode's tensor

The calculation with the Dirac matrices gives:

$$
\begin{aligned}
\overline{\psi}\gamma^0\partial_{\nu}\psi - (\partial_{\nu}\overline{\psi})\gamma^0\psi &= \begin{pmatrix}\xi^{\dagger} & \eta^{\dagger}\end{pmatrix}\begin{pmatrix}\partial_{\nu}\xi \\ \partial_{\nu}\eta\end{pmatrix} - \begin{pmatrix}\partial_{\nu}\xi^{\dagger} & \partial_{\nu}\eta^{\dagger}\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} \\
&= \xi^{\dagger}(\partial_{\nu}\xi) - (\partial_{\nu}\xi^{\dagger})\xi + \eta^{\dagger}(\partial_{\nu}\eta) - (\partial_{\nu}\eta^{\dagger})\eta.
\end{aligned} \tag{B.39}
$$

We thus have:

$$
i\frac{\hbar}{2}c[\overline{\psi}\gamma^0\partial_{\nu}\psi - (\partial_{\nu}\overline{\psi})\gamma^0\psi] = i\frac{\hbar}{2}c[\xi^{\dagger}(\partial_{\nu}\xi) - (\partial_{\nu}\xi^{\dagger})\xi + \eta^{\dagger}(\partial_{\nu}\eta) - (\partial_{\nu}\eta^{\dagger})\eta]. \tag{B.40}
$$

For the γ^j matrices, with $j = 1, 2, 3$ we have:

$$
\begin{aligned}
&\overline{\psi}\gamma^j\partial_{\nu}\psi - (\partial_{\nu}\overline{\psi})\gamma^j\psi \\
&= \begin{pmatrix}\eta^{\dagger} & \xi^{\dagger}\end{pmatrix}\begin{pmatrix}0 & -\sigma_j \\ \sigma_j & 0\end{pmatrix}\begin{pmatrix}\partial_{\nu}\xi \\ \partial_{\nu}\eta\end{pmatrix} - \begin{pmatrix}\partial_{\nu}\eta^{\dagger} & \partial_{\nu}\xi^{\dagger}\end{pmatrix}\begin{pmatrix}0 & -\sigma_j \\ \sigma_j & 0\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} \\
&= \xi^{\dagger}\sigma_j(\partial_{\nu}\xi) - (\partial_{\nu}\xi^{\dagger})\sigma_j\xi - \eta^{\dagger}\sigma_j(\partial_{\nu}\eta) + (\partial_{\nu}\eta^{\dagger})\sigma_j\eta.
\end{aligned} \tag{B.41}
$$

This gives for $j = 1$:

$$
\begin{aligned}
&\overline{\psi}\gamma^1\partial_{\nu}\psi - (\partial_{\nu}\overline{\psi})\gamma^1\psi = \xi^{\dagger}\sigma_1(\partial_{\nu}\xi) - (\partial_{\nu}\xi^{\dagger})\sigma_1\xi - \eta^{\dagger}\sigma_1(\partial_{\nu}\eta) + (\partial_{\nu}\eta^{\dagger})\sigma_1\eta \\
&= \overline{\xi}_2\partial_{\nu}\xi_1 + \overline{\xi}_1\partial_{\nu}\xi_2 - \overline{\eta}_2\partial_{\nu}\eta_1 - \overline{\eta}_1\partial_{\nu}\eta_2 - (\partial_{\nu}\overline{\xi}_2)\xi_1 - (\partial_{\nu}\overline{\xi}_1)\xi_2 \\
&\quad + (\partial_{\nu}\overline{\eta}_2)\eta_1 + (\partial_{\nu}\overline{\eta}_1)\eta_2.
\end{aligned} \tag{B.42}
$$

Similarly we have for $j = 2$:

$$
\begin{aligned}
&\overline{\psi}\gamma^2\partial_{\nu}\psi - (\partial_{\nu}\overline{\psi})\gamma^2\psi = \xi^{\dagger}\sigma_2(\partial_{\nu}\xi) - (\partial_{\nu}\xi^{\dagger})\sigma_2\xi - \eta^{\dagger}\sigma_2(\partial_{\nu}\eta) + (\partial_{\nu}\eta^{\dagger})\sigma_2\eta \\
&= i[\overline{\xi}_2\partial_{\nu}\xi_1 - \overline{\xi}_1\partial_{\nu}\xi_2 - \overline{\eta}_2\partial_{\nu}\eta_1 + \overline{\eta}_1\partial_{\nu}\eta_2 - (\partial_{\nu}\overline{\xi}_2)\xi_1 + (\partial_{\nu}\overline{\xi}_1)\xi_2 \\
&\quad + (\partial_{\nu}\overline{\eta}_2)\eta_1 - (\partial_{\nu}\overline{\eta}_1)\eta_2].
\end{aligned} \tag{B.43}
$$

And for $j = 3$ we get:

$$
\begin{aligned}
&\overline{\psi}\gamma^3\partial_{\nu}\psi - (\partial_{\nu}\overline{\psi})\gamma^3\psi = \xi^{\dagger}\sigma_3(\partial_{\nu}\xi) - (\partial_{\nu}\xi^{\dagger})\sigma_3\xi - \eta^{\dagger}\sigma_3(\partial_{\nu}\eta) + (\partial_{\nu}\eta^{\dagger})\sigma_3\eta \\
&= \overline{\xi}_1\partial_{\nu}\xi_1 - \overline{\xi}_2\partial_{\nu}\xi_2 - \overline{\eta}_1\partial_{\nu}\eta_1 + \overline{\eta}_2\partial_{\nu}\eta_2 - (\partial_{\nu}\overline{\xi}_1)\xi_1 + (\partial_{\nu}\overline{\xi}_2)\xi_2 \\
&\quad + (\partial_{\nu}\overline{\eta}_1)\eta_1 - (\partial_{\nu}\overline{\eta}_2)\eta_2.
\end{aligned} \tag{B.44}
$$

We make now the same calculation with space-time algebra. We have:

$$\partial_\nu \Psi \gamma_{021} \widetilde{\Psi} = \begin{pmatrix} \partial_\nu \phi & 0 \\ 0 & \partial_\nu \widehat{\phi} \end{pmatrix} \begin{pmatrix} 0 & i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix} \begin{pmatrix} \overline{\phi} & 0 \\ 0 & \widetilde{\phi} \end{pmatrix} \tag{B.45}$$

$$= \begin{pmatrix} 0 & i\partial_\nu \phi \sigma_3 \widetilde{\phi} \\ i\partial_\nu \widehat{\phi} \sigma_3 \overline{\phi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & v + iw \\ \widehat{v} - i\widehat{w} & 0 \end{pmatrix}. \tag{B.46}$$

The vectorial part is then:

$$\mathbf{v} = \langle \partial_\nu \Psi \gamma_{021} \widetilde{\Psi} \rangle_1 = \begin{pmatrix} 0 & v \\ \widehat{v} & 0 \end{pmatrix} = v^\mu \gamma_\mu. \tag{B.47}$$

This gives:

$$\begin{aligned} v + iw &= i\partial_\nu \phi \sigma_3 \widetilde{\phi};\ v - iw = (v+iw)^\dagger = -i\phi\sigma_3 \partial_\nu \widetilde{\phi}, \\ v = v^\mu \sigma_\mu &= \frac{i}{2}(\partial_\nu \phi \sigma_3 \widetilde{\phi} - \phi \sigma_3 \partial_\nu \widetilde{\phi}) = \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix} \\ &= i \begin{pmatrix} \begin{matrix} (\partial_\nu \xi_1)\overline{\xi}_1 - (\partial_\nu \overline{\eta}_2)\eta_2 \\ -\xi_1(\partial_\nu \overline{\xi}_1) + \overline{\eta}_2(\partial_\nu \eta_2) \end{matrix} & \begin{matrix} (\partial_\nu \xi_1)\overline{\xi}_2 + (\partial_\nu \overline{\eta}_2)\eta_1 \\ -\xi_1(\partial_\nu \overline{\xi}_2) - \overline{\eta}_2(\partial_\nu \eta_1) \end{matrix} \\ \\ \begin{matrix} (\partial_\nu \xi_2)\overline{\xi}_1 + (\partial_\nu \overline{\eta}_1)\eta_2 \\ -\xi_2(\partial_\nu \overline{\xi}_1) - \overline{\eta}_1(\partial_\nu \eta_2) \end{matrix} & \begin{matrix} (\partial_\nu \xi_2)\overline{\xi}_2 - (\partial_\nu \overline{\eta}_1)\eta_1 \\ -\xi_2(\partial_\nu \overline{\xi}_2) + \overline{\eta}_1(\partial_\nu \eta_1) \end{matrix} \end{pmatrix}. \end{aligned} \tag{B.48}$$

We then get:

$$\begin{aligned} v^0 + v^3 &= i[(\partial_\nu \xi_1)\overline{\xi}_1 - (\partial_\nu \overline{\eta}_2)\eta_2 - \xi_1(\partial_\nu \overline{\xi}_1) + \overline{\eta}_2(\partial_\nu \eta_2)], \\ v^0 - v^3 &= i[(\partial_\nu \xi_2)\overline{\xi}_2 - (\partial_\nu \overline{\eta}_1)\eta_1 - \xi_2(\partial_\nu \overline{\xi}_2) + \overline{\eta}_1(\partial_\nu \eta_1)], \\ v^1 + iv^2 &= i[(\partial_\nu \xi_2)\overline{\xi}_1 + (\partial_\nu \overline{\eta}_1)\eta_2 - \xi_2(\partial_\nu \overline{\xi}_1) - \overline{\eta}_1(\partial_\nu \eta_2)], \\ v^1 - iv^2 &= i[(\partial_\nu \xi_1)\overline{\xi}_2 + (\partial_\nu \overline{\eta}_2)\eta_1 - \xi_1(\partial_\nu \overline{\xi}_2) - \overline{\eta}_2(\partial_\nu \eta_1)]. \end{aligned} \tag{B.49}$$

By adding and subtracting, this gives:

$$\begin{aligned} v^0 &= \frac{i}{2}[\xi^\dagger \partial_\nu \xi - (\partial_\nu \xi^\dagger)\xi + \eta^\dagger \partial_\nu \eta - (\partial_\nu \eta^\dagger)\eta] = \gamma^0 \cdot \langle \partial_\nu \Psi \gamma_{021} \widetilde{\Psi} \rangle_1, \\ v^3 &= \frac{i}{2}[\xi^\dagger \sigma_3 \partial_\nu \xi - (\partial_\nu \xi^\dagger)\sigma_3 \xi - \eta^\dagger \sigma_3 \partial_\nu \eta 2(\partial_\nu \eta^\dagger)\sigma_3 \eta] = \gamma^3 \cdot \langle \partial_\nu \Psi \gamma_{021} \widetilde{\Psi} \rangle_1, \\ v^1 &= \frac{i}{2}[\xi^\dagger \sigma_1 \partial_\nu \xi - (\partial_\nu \xi^\dagger)\sigma_1 \xi - \eta^\dagger \sigma_1 \partial_\nu \eta 2(\partial_\nu \eta^\dagger)\sigma_1 \eta] = \gamma^1 \cdot \langle \partial_\nu \Psi \gamma_{021} \widetilde{\Psi} \rangle_1, \\ v^2 &= \frac{i}{2}[\xi^\dagger \sigma_2 \partial_\nu \xi - (\partial_\nu \xi^\dagger)\sigma_2 \xi - \eta^\dagger \sigma_2 \partial_\nu \eta 2(\partial_\nu \eta^\dagger)\sigma_2 \eta] = \gamma^2 \cdot \langle \partial_\nu \Psi \gamma_{021} \widetilde{\Psi} \rangle_1. \end{aligned} \tag{B.50}$$

B.2 General and reverse terms in $Cl_{1,5}$ and $Cl_{3,3}$

We previously used the Clifford algebra $Cl_{1,5}$ as a natural generalization of $Cl_{1,3}$, the space-time algebra of Hestenes' works [73]–[78]. We may link the $Cl_{1,3}$ algebra to this greater algebra by using, with $\mu = 0, 1, 2, 3$:

$$L_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix} \; ; \; L_4 = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix} \; ; \; L_5 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \tag{B.51}$$

where I_4 is the unit 4×4 matrix and where:

$$\mathbf{i} = \gamma_{0123} = \gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma_5; \; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{B.52}$$

We use (1.4) and we recall:

$$\gamma_0 = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \; \gamma^j = -\gamma_j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}; \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where the σ_j are the Pauli matrices. The $Cl_{3,3}$ algebra is isomorphic to the $M_8(\mathbb{R})$ algebra, and thus to the algebra of the endomorphisms in Cl_3. This End(Cl_3) algebra is linked to $Cl_{1,3}$ and $Cl_{1,5}$ by:

$$\Gamma_\mu = L_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}, \; \mu = 0, 1, 2, 3, \tag{B.53}$$

$$\Gamma_4 = iL_4 = \begin{pmatrix} 0 & -iI_4 \\ iI_4 & 0 \end{pmatrix} \; ; \; \Gamma_5 = -iL_5 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \tag{B.54}$$

The indices $\mu, \nu, \rho \ldots$ are $0, 1, 2, 3$ and the indices a, b, c, d, e are $0, 1, 2, 3, 4, 5$. We have:

$$\Gamma_{\mu\nu} = L_{\mu\nu} = L_\mu L_\nu = \begin{pmatrix} \gamma_{\mu\nu} & 0 \\ 0 & \gamma_{\mu\nu} \end{pmatrix}, \tag{B.55}$$

$$\Gamma_{\mu\nu\rho} = L_{\mu\nu\rho} = L_{\mu\nu} L_\rho = \begin{pmatrix} 0 & \gamma_{\mu\nu\rho} \\ \gamma_{\mu\nu\rho} & 0 \end{pmatrix}, \tag{B.56}$$

$$\Gamma_{0123} = L_{0123} = L_{01} L_{23} = \begin{pmatrix} \gamma_{0123} & 0 \\ 0 & \gamma_{0123} \end{pmatrix} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}, \tag{B.57}$$

$$\Gamma_{45} = L_{45} = L_4 L_5 = \begin{pmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}, \tag{B.58}$$

$$\Gamma_{012345} = L_{012345} = L_{0123} L_{45} = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}. \tag{B.59}$$

We also get:

$$L_{01235} = L_{0123} L_5 = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -I_4 \\ -I_4 & 0 \end{pmatrix}, \tag{B.60}$$

$$L_{\mu 4} = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{pmatrix} \; ; \; L_{\mu 5} = \begin{pmatrix} \gamma_\mu \mathbf{i} & 0 \\ 0 & \gamma_\mu \mathbf{i} \end{pmatrix}, \tag{B.61}$$

$$L_{\mu\nu 4} = \begin{pmatrix} 0 & -\gamma_{\mu\nu} \\ \gamma_{\mu\nu} & 0 \end{pmatrix} \; ; \; L_{\mu\nu 5} = \begin{pmatrix} 0 & \gamma_{\mu\nu}\mathbf{i} \\ \gamma_{\mu\nu}\mathbf{i} & 0 \end{pmatrix}, \tag{B.62}$$

$$L_{\mu\nu\rho 4} = \begin{pmatrix} \gamma_{\mu\nu\rho} & 0 \\ 0 & -\gamma_{\mu\nu\rho} \end{pmatrix} \; ; \; L_{\mu\nu\rho 5} = \begin{pmatrix} \gamma_{\mu\nu\rho}\mathbf{i} & 0 \\ 0 & \gamma_{\mu\nu\rho}\mathbf{i} \end{pmatrix}. \tag{B.63}$$

Similarly we get:

$$L_{\mu 45} = \begin{pmatrix} 0 & \gamma_\mu \mathbf{i} \\ -\gamma_\mu \mathbf{i} & 0 \end{pmatrix} \ ; \ L_{\mu\nu 45} = \begin{pmatrix} -\gamma_{\mu\nu} \mathbf{i} & 0 \\ 0 & \gamma_{\mu\nu} \mathbf{i} \end{pmatrix}, \tag{B.64}$$

$$L_{\mu\nu\rho 45} = \begin{pmatrix} 0 & \gamma_{\mu\nu\rho} \mathbf{i} \\ -\gamma_{\mu\nu\rho} \mathbf{i} & 0 \end{pmatrix} \ ; \ L_{01234} = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}. \tag{B.65}$$

The general term in $Cl_{1,5}$ can be expressed as:

$$\Psi^{1,5} = \Psi_0^{1,5} + \Psi_1^{1,5} + \Psi_2^{1,5} + \Psi_3^{1,5} + \Psi_4^{1,5} + \Psi_5^{1,5} + \Psi_6^{1,5}, \tag{B.66}$$

$$\Psi_1^{1,5} = \sum_{a=0}^{a=5} N^a L_a,\ \Psi_2^{1,5} = \sum_{0\leqslant a<b\leqslant 5} N^{ab} L_{ab},\ \Psi_3^{1,5} = \sum_{0\leqslant a<b<c\leqslant 5} N^{abc} L_{abc},$$

$$\Psi_4^{1,5} = \sum_{0\leqslant a<b<c<d\leqslant 5} N^{abcd} L_{abcd},\ \Psi_5^{1,5} = \sum_{0\leqslant a<b<c<d<e\leqslant 5} N^{abcde} L_{abcde},$$

$$\Psi_0^{1,5} = sI_8, s \in \mathbb{R} \ ; \quad \Psi_6^{1,5} = pL_{012345},\ p \in \mathbb{R}. \tag{B.67}$$

where N^{ind} are real numbers. The general term in $Cl_{3,3}$ is expressed as:

$$\Psi^{3,3} = \Psi_0^{3,3} + \Psi_1^{3,3} + \Psi_2^{3,3} + \Psi_3^{3,3} + \Psi_4^{3,3} + \Psi_5^{3,3} + \Psi_6^{3,3}, \tag{B.68}$$

$$\Psi_1^{3,3} = \sum_{a=0}^{a=5} N^a \Gamma_a,\ \Psi_2^{3,3} = \sum_{0\leqslant a<b\leqslant 5} N^{ab} \Gamma_{ab},\ \Psi_3^{3,3} = \sum_{0\leqslant a<b<c\leqslant 5} N^{abc} \Gamma_{abc},$$

$$\Psi_4^{3,3} = \sum_{0\leqslant a<b<c<d\leqslant 5} N^{abcd} \Gamma_{abcd},\ \Psi_5^{3,3} = \sum_{0\leqslant a<b<c<d<e\leqslant 5} N^{abcde} \Gamma_{abcde},$$

$$\Psi_0^{3,3} = sI_8, s \in \mathbb{R} \ ; \quad \Psi_6^{3,3} = p\Gamma_{012345} = pL_{012345},\ p \in \mathbb{R}. \tag{B.69}$$

With $Cl_{3,3}$ we have:

$$\Gamma_{ind\,4} = iL_{ind\,4};\ \Gamma_{ind\,5} = -iL_{ind\,5};\ \Gamma_{ind\,45} = L_{ind\,45}. \tag{B.70}$$

The scalar and pseudoscalar terms have the following form in both algebras:

$$\alpha I_8 + \omega L_{012345} = \begin{pmatrix} (\alpha+\omega) I_4 & 0 \\ 0 & (\alpha-\omega) I_4 \end{pmatrix}, \tag{B.71}$$

$$\alpha I_8 - \omega L_{012345} = \begin{pmatrix} (\alpha-\omega) I_4 & 0 \\ 0 & (\alpha+\omega) I_4 \end{pmatrix}. \tag{B.72}$$

For the calculation of the 1-vector term:

$$N^a L_a = N^4 L_4 + N^5 L_5 + N^\mu L_\mu,$$

we let:

$$\beta = N^4 \ ; \quad \delta = N^5 \ ; \quad \mathbf{a} = N^\mu \gamma_\mu. \tag{B.73}$$

This gives:

$$\Psi_1^{1,5} = \begin{pmatrix} 0 & -\beta I_4 + \delta\mathbf{i} + \mathbf{a} \\ \beta I_4 + \delta\mathbf{i} + \mathbf{a} & 0 \end{pmatrix}, \tag{B.74}$$

$$\Psi_1^{3,3} = \begin{pmatrix} 0 & -i\beta I_4 - i\delta\mathbf{i} + \mathbf{a} \\ i\beta I_4 - i\delta\mathbf{i} + \mathbf{a} & 0 \end{pmatrix}. \tag{B.75}$$

For the calculation of the 2-vector term:

$$N^{ab}L_{ab} = N^{45}L_{45} + N^{\mu 4}L_{\mu 4} + N^{\mu 5}L_{\mu 5} + N^{\mu\nu}L_{\mu\nu},$$

we let:

$$\epsilon = N^{45} \ ; \quad \mathbf{b} = N^{\mu 4}\gamma_\mu \ ; \quad \mathbf{c} = N^{\mu 5}\gamma_\mu \ ; \quad \mathbf{A} = N^{\mu\nu}\gamma_{\mu\nu}. \tag{B.76}$$

This gives:

$$\Psi_2^{1,5} = \begin{pmatrix} -\epsilon\mathbf{i} + \mathbf{b} - \mathbf{ic} + \mathbf{A} & 0 \\ 0 & \epsilon\mathbf{i} - \mathbf{b} - \mathbf{ic} + \mathbf{A} \end{pmatrix}, \tag{B.77}$$

$$\Psi_2^{3,3} = \begin{pmatrix} -\epsilon\mathbf{i} + i\mathbf{b} + i\mathbf{ic} + \mathbf{A} & 0 \\ 0 & \epsilon\mathbf{i} - i\mathbf{b} + i\mathbf{ic} + \mathbf{A} \end{pmatrix}. \tag{B.78}$$

For the calculation of the 3-vector term:

$$N^{abc}L_{abc} = N^{\mu 45}L_{\mu 45} + N^{\mu\nu 4}L_{\mu\nu 4} + N^{\mu\nu 5}L_{\mu\nu 5} + N^{\mu\nu\rho}L_{\mu\nu\rho},$$

we let:

$$\mathbf{d} = N^{\mu 45}\gamma_\mu \ ; \quad \mathbf{B} = N^{\mu\nu 4}\gamma_{\mu\nu} \ ; \quad \mathbf{C} = N^{\mu\nu 5}\gamma_{\mu\nu} \ ; \quad \mathbf{ie} = N^{\mu\nu\rho}\gamma_{\mu\nu\rho}. \tag{B.79}$$

This gives:

$$\Psi_3^{1,5} = \begin{pmatrix} 0 & \mathbf{di} - \mathbf{B} + \mathbf{iC} + \mathbf{ie} \\ \mathbf{id} + \mathbf{B} + \mathbf{iC} + \mathbf{ie} & 0 \end{pmatrix}, \tag{B.80}$$

$$\Psi_3^{3,3} = \begin{pmatrix} 0 & \mathbf{di} - i\mathbf{B} - i\mathbf{iC} + \mathbf{ie} \\ \mathbf{id} + i\mathbf{B} - i\mathbf{iC} + \mathbf{ie} & 0 \end{pmatrix}. \tag{B.81}$$

For the calculation of the 4-vector term:

$$N^{abcd}L_{abcd} = N^{\mu\nu 45}L_{\mu\nu 45} + N^{\mu\nu\rho 4}L_{\mu\nu\rho 4} + N^{\mu\nu\rho 5}L_{\mu\nu\rho 5} + N^{0123}L_{0123},$$

we let:

$$\mathbf{D} = N^{\mu\nu 45}\gamma_{\mu\nu} \ ; \quad \mathbf{if} = N^{\mu\nu\rho 4}\gamma_{\mu\nu\rho} \ ; \quad \mathbf{ig} = N^{\mu\nu\rho 5}\gamma_{\mu\nu\rho} \ ; \quad \zeta = N^{0123}. \tag{B.82}$$

This gives:

$$\Psi_4^{1,5} = \begin{pmatrix} -\mathbf{iD} + \mathbf{if} + \mathbf{g} + \zeta\mathbf{i} & 0 \\ 0 & \mathbf{iD} - \mathbf{if} + \mathbf{g} + \zeta\mathbf{i} \end{pmatrix}, \tag{B.83}$$

$$\Psi_4^{3,3} = \begin{pmatrix} -\mathbf{iD} + i\mathbf{if} - i\mathbf{g} + \zeta\mathbf{i} & 0 \\ 0 & \mathbf{iD} - i\mathbf{if} - i\mathbf{g} + \zeta\mathbf{i} \end{pmatrix}. \tag{B.84}$$

For the calculation of the 5-vector term:

$$N^{abcde}L_{abcde} = N^{\mu\nu\rho 45}L_{\mu\nu\rho 45} + N^{01234}L_{01234} + N^{01235}L_{01235},$$

we let:

$$\mathbf{ih} = N^{\mu\nu\rho 45}\gamma_{\mu\nu\rho}\ ; \quad \eta = N^{01234}\ ; \quad \theta = N^{01235}. \tag{B.85}$$

This gives:

$$\Psi_5^{1,5} = \begin{pmatrix} 0 & \mathbf{h} - \eta\mathbf{i} - \theta I_4 \\ -\mathbf{h} + \eta\mathbf{i} - \theta I_4 & \end{pmatrix}, \tag{B.86}$$

$$\Psi_5^{3,3} = \begin{pmatrix} 0 & \mathbf{h} - i\eta\mathbf{i} + i\theta I_4 \\ -\mathbf{h} + i\eta\mathbf{i} + i\theta I_4 & \end{pmatrix}. \tag{B.87}$$

We then get:

$$\Psi^{1,5} = \begin{pmatrix} \Psi_1 & \Psi_2 \\ \Psi_3 & \Psi_4 \end{pmatrix} \tag{B.88}$$

$$\begin{aligned}
\Psi_1 &= (\alpha + \omega) + (\mathbf{b} + \mathbf{g}) + (\mathbf{A} - \mathbf{iD}) + \mathbf{i}(-\mathbf{c} + \mathbf{f}) + (\zeta - \epsilon)\mathbf{i}, \\
\Psi_2 &= -(\beta + \theta) + (\mathbf{a} + \mathbf{h}) + (-\mathbf{B} + \mathbf{iC}) + \mathbf{i}(-\mathbf{d} + \mathbf{e}) + (\delta - \eta)\mathbf{i}, \\
\Psi_3 &= (\beta - \theta) + (\mathbf{a} - \mathbf{h}) + (\mathbf{B} + \mathbf{iC}) + \mathbf{i}(\mathbf{d} + \mathbf{e}) + (\delta + \eta)\mathbf{i}, \\
\Psi_4 &= (\alpha - \omega) + (-\mathbf{b} + \mathbf{g}) + (\mathbf{A} + \mathbf{iD}) + \mathbf{i}(-\mathbf{c} - \mathbf{f}) + (\zeta + \epsilon)\mathbf{i}.
\end{aligned} \tag{B.89}$$

Thus we have:

$$\frac{1}{2}(\Psi_1 + \Psi_4) = \mathcal{P}_1 + \mathcal{I}_1;\ \mathcal{P}_1 = \alpha + \mathbf{A} + \zeta\mathbf{i};\ \mathcal{I}_1 = \mathbf{g} - \mathbf{ic}, \tag{B.90}$$

$$\frac{1}{2}(\Psi_1 - \Psi_4) = \mathcal{P}_4 + \mathcal{I}_4;\ \mathcal{P}_4 = \omega - \mathbf{iD} - \epsilon\mathbf{i};\ \mathcal{I}_4 = \mathbf{b} + \mathbf{if}, \tag{B.91}$$

$$\frac{1}{2}(\Psi_2 + \Psi_3) = \mathcal{P}_2 + \mathcal{I}_2;\ \mathcal{P}_2 = -\theta + \mathbf{iC} + \delta\mathbf{i};\ \mathcal{I}_2 = \mathbf{a} + \mathbf{ie}, \tag{B.92}$$

$$\frac{1}{2}(-\Psi_2 + \Psi_3) = \mathcal{P}_3 - \mathcal{I}_3;\ \mathcal{P}_3 = \beta + \mathbf{B} + \eta\mathbf{i};\ \mathcal{I}_3 = \mathbf{h} - \mathbf{id}. \tag{B.93}$$

The general term in $Cl_{3,3}$ is:

$$\Psi^{3,3} = \begin{pmatrix} \Psi_l + i\Psi_b & \Psi_r + \Psi_g \\ \Psi_r - \Psi_g & \Psi_l - i\Psi_b \end{pmatrix}, \tag{B.94}$$

$$\begin{aligned}
\Psi_l + i\Psi_b &= \alpha + \mathbf{A} + \zeta\mathbf{i} - i(\mathbf{g} - \mathbf{ic}) + \omega - \mathbf{iD} - \epsilon\mathbf{i} + i(\mathbf{b} + \mathbf{if}), \\
\Psi_l - i\Psi_b &= \alpha + \mathbf{A} + \zeta\mathbf{i} - i(\mathbf{g} - \mathbf{ic}) - [\omega - \mathbf{iD} - \epsilon\mathbf{i} + i(\mathbf{b} + \mathbf{if})], \\
\Psi_r + \Psi_g &= -i(-\theta + \mathbf{iC} + \delta\mathbf{i}) + \mathbf{a} + \mathbf{ie} - i(\beta + \mathbf{B} + \eta\mathbf{i}) + \mathbf{h} - \mathbf{id}, \\
\Psi_r - \Psi_g &= -i(-\theta + \mathbf{iC} + \delta\mathbf{i}) + \mathbf{a} + \mathbf{ie} + i(\beta + \mathbf{B} + \eta\mathbf{i}) - (\mathbf{h} - \mathbf{id}).
\end{aligned} \tag{B.95}$$

This gives:

$$\Psi_l = \mathcal{P}_1 - i\mathcal{I}_1;\ \mathcal{P}_1 = \begin{pmatrix} \phi_e & 0 \\ 0 & \widehat{\phi}_e \end{pmatrix} = \alpha + \mathbf{A} + \zeta\mathbf{i}, \tag{B.96}$$

$$\mathcal{I}_1 = \begin{pmatrix} 0 & \phi_n \\ \widehat{\phi}_n & 0 \end{pmatrix} = \mathbf{g} - \mathbf{ic}, \tag{B.97}$$

$$\Psi_r = -i\mathcal{P}_2 + \mathcal{I}_2;\ \mathcal{P}_2 = \begin{pmatrix} \phi_{dr} & 0 \\ 0 & \widehat{\phi}_{dr} \end{pmatrix} = -\theta + \mathbf{iC} + \delta\mathbf{i}, \tag{B.98}$$

$$\mathcal{I}_2 = \begin{pmatrix} 0 & \phi_{ur} \\ \widehat{\phi}_{ur} & 0 \end{pmatrix} = \mathbf{a} + \mathbf{ie}, \tag{B.99}$$

$$\Psi_g = -i\mathcal{P}_3 + \mathcal{I}_3;\ \mathcal{P}_3 = \begin{pmatrix} \phi_{dg} & 0 \\ 0 & \widehat{\phi}_{dg} \end{pmatrix} = \beta + \mathbf{B} + \eta\mathbf{i}, \tag{B.100}$$

$$\mathcal{I}_3 = \begin{pmatrix} 0 & \phi_{ug} \\ \widehat{\phi}_{ug} & 0 \end{pmatrix} = \mathbf{h} - \mathbf{id}, \tag{B.101}$$

$$\Psi_b = -i\mathcal{P}_4 + \mathcal{I}_4;\ \mathcal{P}_4 = \begin{pmatrix} \phi_{db} & 0 \\ 0 & \widehat{\phi}_{db} \end{pmatrix} = \omega - \mathbf{iD} - \epsilon\mathbf{i}, \tag{B.102}$$

$$\mathcal{I}_4 = \begin{pmatrix} 0 & \phi_{ub} \\ \widehat{\phi}_{ub} & 0 \end{pmatrix} = \mathbf{b} + \mathbf{if}. \tag{B.103}$$

Here Ψ_l is alone while Ψ_r, Ψ_g and Ψ_b have exactly the same structure: **this is the origin of the difference between lepton and quark waves**.

In $Cl_{1,3}$ the reverse of $A = A_0 + A_1 + A_2 + A_3 + A_4$ is $\widetilde{A} = A_0 + A_1 - A_2 - A_3 + A_4$, we must change the sign of the bivectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{iC}$, $\mathbf{iD}$, and the 3-vectors $\mathbf{ic}$, $\mathbf{id}$, $\mathbf{ie}$, $\mathbf{if}$ and we then get:

$$\widetilde{\Psi}_l = \widetilde{\mathcal{P}}_1 - i\widetilde{\mathcal{I}}_1;\ \widetilde{\mathcal{P}}_1 = \begin{pmatrix} \overline{\phi}_e & 0 \\ 0 & \phi_e^\dagger \end{pmatrix} = \alpha - \mathbf{A} + \zeta\mathbf{i}, \tag{B.104}$$

$$\widetilde{\mathcal{I}}_1 = \begin{pmatrix} 0 & \phi_n^\dagger \\ \overline{\phi}_n & 0 \end{pmatrix} = \mathbf{g} + \mathbf{ic}, \tag{B.105}$$

$$\widetilde{\Psi}_r = -i\widetilde{\mathcal{P}}_2 + \widetilde{\mathcal{I}}_2;\ \widetilde{\mathcal{P}}_2 = \begin{pmatrix} \overline{\phi}_{dr} & 0 \\ 0 & \phi_{dr}^\dagger \end{pmatrix} = -\theta - \mathbf{iC} + \delta\mathbf{i}, \tag{B.106}$$

$$\widetilde{\mathcal{I}}_2 = \begin{pmatrix} 0 & \phi_{ur} \\ \overline{\phi}_{ur} & 0 \end{pmatrix} = \mathbf{a} - \mathbf{ie}, \tag{B.107}$$

$$\widetilde{\Psi}_g = -i\widetilde{\mathcal{P}}_3 + \widetilde{\mathcal{I}}_3;\ \widetilde{\mathcal{P}}_3 = \begin{pmatrix} \overline{\phi}_{dg} & 0 \\ 0 & \phi_{dg}^\dagger \end{pmatrix} = \beta - \mathbf{B} + \eta\mathbf{i}, \tag{B.108}$$

$$\widetilde{\mathcal{I}}_3 = \begin{pmatrix} 0 & \phi_{ug} \\ \overline{\phi}_{ug} & 0 \end{pmatrix} = \mathbf{h} + \mathbf{id}, \tag{B.109}$$

$$\widetilde{\Psi}_b = -i\widetilde{\mathcal{P}}_4 + \widetilde{\mathcal{I}}_4;\ \widetilde{\mathcal{P}}_4 = \begin{pmatrix} \overline{\phi}_{db} & 0 \\ 0 & \phi^\dagger_{db} \end{pmatrix} = \omega + \mathbf{iD} - \epsilon\mathbf{i}, \tag{B.110}$$

$$\widetilde{\mathcal{I}}_4 = \begin{pmatrix} 0 & \phi_{ub} \\ \overline{\phi}_{ub} & 0 \end{pmatrix} = \mathbf{b} - \mathbf{if}. \tag{B.111}$$

Now the reverse in $Cl_{3,3}$ of

$$A = A_0 + A_1 + A_2 + A_3 + A_4 + A_5 + A_6,$$

is

$$\widetilde{A} = A_0 + A_1 - A_2 - A_3 + A_4 + A_5 - A_6.$$

The only terms that change signs are the ϵ and ω scalars, the $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$ vectors and the $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ bivectors. These changes of sign are not the same in $Cl_{3,3}$ and in $Cl_{1,3}$. The differences are compensated for by the swapping made in $Cl_{3,3}$ between the positions of the Ψ_l and Ψ_b terms. Hence the $\mathcal{P}_n$ are transformed into $\widetilde{\mathcal{P}}_n$ and the $\mathcal{I}_n$ are transformed into $\widetilde{\mathcal{I}}_n$ for $n = 1, 2, 3$. On the other hand $\mathcal{P}_4$ is changed into $-\widetilde{\mathcal{P}}_4$, and $\mathcal{I}_4$ is changed into $-\widetilde{\mathcal{I}}_4$. We thus get:

$$\widetilde{\Psi}^{3,3} = \begin{pmatrix} \widetilde{\Psi}_l - i\widetilde{\Psi}_b & \widetilde{\Psi}_r + \widetilde{\Psi}_g \\ \widetilde{\Psi}_r - \widetilde{\Psi}_g & \widetilde{\Psi}_l + i\widetilde{\Psi}_b \end{pmatrix}. \tag{B.112}$$

With:

$$\Psi_l = \begin{pmatrix} \phi_e & -i\phi_n \\ -i\widehat{\phi}_n & \widehat{\phi}_e \end{pmatrix};\ \widetilde{\Psi}_l = \begin{pmatrix} \overline{\phi}_e & -i\phi^\dagger_n \\ -i\overline{\phi}_n & \phi^\dagger_e \end{pmatrix}, \tag{B.113}$$

$$\Psi_c = \begin{pmatrix} -i\phi_{dc} & \phi_{uc} \\ \widehat{\phi}_{uc} & -i\widehat{\phi}_{dc} \end{pmatrix};\ \widetilde{\Psi}_c = \begin{pmatrix} -i\overline{\phi}_{dc} & \phi^\dagger_{uc} \\ \overline{\phi}_{uc} & -i\phi^\dagger_{dc} \end{pmatrix},\ c = r, g, b. \tag{B.114}$$

Since $\mathcal{P}_1 = \alpha + \mathbf{A} + \zeta\mathbf{i}$ is the general element of either $Cl^+_{1,3}$ or $Cl^+_{3,1}$, since $\mathcal{I}_1 = \mathbf{g} - \mathbf{ic}$ is the general odd element of $Cl_{1,3}$ while $-i\mathcal{I}_1$ is the general odd element of $Cl_{3,1}$, thus $\Psi_l = \mathcal{P}_1 - i\mathcal{I}_1$ is the general element of $Cl_{3,1}$. Moreover $i\Psi_r = \mathcal{P}_2 + i\mathcal{I}_2$, $i\Psi_g = \mathcal{P}_3 + i\mathcal{I}_3$ and $i\Psi_b = \mathcal{P}_4 + i\mathcal{I}_4$ imply that the three $i\Psi_c$ are general elements of $Cl_{3,1}$. It is noticeable that the three Ψ_c do not have the properties of Ψ_l, which is similar to the three $i\Psi_c$, $c = r, g, b$. The well-known equalities $Cl_{3,1} = M_4(\mathbb{R})$ and $Cl_{3,3} = M_8(\mathbb{R})$ lead us to calculate the general element of $Cl_{3,3}$ from four blocks in $Cl_{3,1}$ but here we get:

$$M = \begin{pmatrix} A & iB \\ iC & D \end{pmatrix}, A, B, C, D \in Cl_{3,1}. \tag{B.115}$$

This does not change the calculations with blocks, since i commutes with the four blocks.

Appendix C

The Hydrogen Atom

We present the resolution of our improved equation for the hydrogen atom. Our resolution uses a method separating the variables in spherical coordinates. Angular functions use Gegenbauer's polynomial functions previously used in linear Dirac theory. Here we study new solutions, for an electron with both electric charge and left and right masses.

The hydrogen atom is the jewel of Dirac theory. The first solutions calculated by C. G. Darwin [11], which we may also find in newer reports [99], are proper values of an *ad hoc* operator, obtained from the nonrelativistic theory, which is not the total angular momentum operator. These solutions give the expected number of states, the true formula for the energy levels, and have the expected nonrelativistic approximations. This was considered very satisfactory. But most of Darwin's solutions suffer the disadvantage that they have a Yvon–Takabayasi angle that is not everywhere defined and small. Therefore they cannot be linear approximations of the solutions to our improved equation.

We previously obtained [14] new solutions in the linear case which have a Yvon–Takabayasi angle everywhere defined and small, and so those may be the linear approximations of the solutions for our nonlinear wave equation. Here we do not use these approximations: we study the actual solutions of our improved nonlinear equation.

C.1 Separating variables

To solve the Dirac equation or the improved equation in the case of the hydrogen atom, two methods exist. Here we shall use, not the initial method based on the nonrelativistic wave equations, but the new method invented more recently by H. Krüger [81], a very fine classic method from the mathematical point of view for a partial differential equation, separating the variables in spherical coordinates:

$$x^1 =: r\sin\theta\cos\varphi \ \ ; \ \ x^2 =: r\sin\theta\sin\varphi \ \ ; \ \ x^3 =: r\cos\theta. \tag{C.1}$$

We use the following notations [1]:

$$i_1 = \sigma_{23} := i\sigma_1 \ ; \ \ i_2 = \sigma_{31} := i\sigma_2 \ ; \ \ i_3 = \sigma_{12} := i\sigma_3, \tag{C.2}$$

$$S := e^{-\frac{\varphi}{2}i_3}e^{-\frac{\theta}{2}i_2} \ ; \ \ \Omega = \widehat{\Omega} := r^{-1}(\sin\theta)^{-\frac{1}{2}}S, \tag{C.3}$$

$$\vec{\partial}' := \sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi.$$

H. Krüger obtained the remarkable identity:

$$\vec{\partial} = \Omega\vec{\partial}'\Omega^{-1}, \tag{C.4}$$

which with:

$$\nabla' := \partial_0 - \vec{\partial}' = \partial_0 - \Big(\sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi\Big), \tag{C.5}$$

also yields:

$$\Omega^{-1}\nabla = \nabla'\Omega^{-1}. \tag{C.6}$$

Here we solve the improved wave equation in its form taken with a left wave and a right wave. The equations that we have to resolve are thus:

$$\begin{aligned}
&0 = \nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + \mathrm{v}\widehat{\phi}\mathbf{m};\ \widehat{\phi} = \sqrt{2}(\eta^1 \ \ \widehat{\xi}^1),\\
&0 = (-i\nabla + qA + \mathbf{l}\mathrm{v})\eta^1;\ 0 = (-i\nabla + qA)\eta^1 + \mathbf{l}e^{-i\beta}\xi^1,\\
&0 = (i\nabla + qA + \mathbf{r}\mathrm{v})\widehat{\xi}^1;\ 0 = (i\nabla + qA)\widehat{\xi}^1 + \mathbf{r}e^{-i\beta}\widehat{\eta}^1.
\end{aligned} \tag{C.7}$$

For the Dirac equation or the improved equation, with the goal of separating the temporal variable $x^0 := ct$ and the angular variable φ from the radial variable r and the other angular variable θ, we let:

$$\phi =: \Omega X e^{(\lambda\varphi - Ex^0 + \delta)i_3};\ X =: (\xi \ \ \widehat{\eta}), \tag{C.8}$$

where X is a function, with value in the Pauli algebra, of only r and θ, $\hbar cE$ is the energy of the electron, and δ is an arbitrary phase that plays no role here because the wave equations are electric gauge-invariant. λ is a real constant which will be interpreted as the magnetic quantum number. We then have:

$$\Omega^{-1}\phi = Xe^{(\lambda\varphi - Ex^0 + \delta)i_3},$$

$$\Omega^{-1}\widehat{\phi} = \widehat{X}e^{(\lambda\varphi - Ex^0 + \delta)i_3} \tag{C.9}$$

$$= (e^{i(\lambda\varphi - Ex^0 + \delta)}\eta \ \ e^{-i(\lambda\varphi - Ex^0 + \delta)}\widehat{\xi}). \tag{C.10}$$

1. S has nothing to do with the tensor S_3, and Ω must not be confused with the relativistic invariants Ω_1 and Ω_2 studied in Chapter 1.

We also have:

$$\rho e^{i\beta} = \det(\phi) = \det(\Omega)\det(X)\det[e^{(\lambda\varphi - Ex^0+\delta)i_3}],$$
$$\det(\Omega) = r^{-2}(\sin\theta)^{-1}\ ;\quad \det[e^{(\lambda\varphi - Ex^0+\delta)i_3}] = 1,$$
$$\rho e^{i\beta} = \frac{\det(X)}{r^2\sin\theta}. \tag{C.11}$$

Thus if we let:

$$\rho_X e^{i\beta_X} =: \det(X), \tag{C.12}$$

we get:

$$\rho = \frac{\rho_X}{r^2\sin\theta}\ ;\quad \beta = \beta_X. \tag{C.13}$$

Hence with the form (C.8) of the wave, the Yvon-Takabayasi angle depends neither on time nor on the φ angle. It depends only on r and θ. It is why the separation of variables can begin similarly for either the Dirac equation or the improved equation. We have:

$$\nabla'\Omega^{-1}\widehat{\phi} = (\partial_0 - \sigma_3\partial_r - \frac{1}{r}\sigma_1\partial_\theta - \frac{1}{r\sin\theta}\sigma_2\partial_\varphi)[\widehat{X}e^{(\lambda\varphi - Ex^0+\delta)i_3}], \tag{C.14}$$
$$\partial_0[\widehat{X}e^{(\lambda\varphi - Ex^0+\delta)i_3}] = -E\widehat{X}i_3 e^{(\lambda\varphi - Ex^0+\delta)i_3}, \tag{C.15}$$
$$\partial_r[\widehat{X}e^{(\lambda\varphi - Ex^0+\delta)i_3}] = (\partial_r\widehat{X})e^{(\lambda\varphi - Ex^0+\delta)i_3} \tag{C.16}$$
$$\partial_\theta[\widehat{X}e^{(\lambda\varphi - Ex^0+\delta)i_3}] = (\partial_\theta\widehat{X})e^{(\lambda\varphi - Ex^0+\delta)i_3}, \tag{C.17}$$
$$\partial_\varphi[\widehat{X}e^{(\lambda\varphi - Ex^0+\delta)i_3}] = \lambda\widehat{X}i_3 e^{(\lambda\varphi - Ex^0+\delta)i_3}. \tag{C.18}$$

We thus get:

$$\nabla\widehat{\phi} = \Omega\Big(-E\widehat{X}i_3 - \sigma_3\partial_r\widehat{X} - \frac{1}{r}\sigma_1\partial_\theta\widehat{X} - \frac{\lambda}{r\sin\theta}\sigma_2\widehat{X}i_3\Big)e^{(\lambda\varphi - Ex^0+\delta)i_3}. \tag{C.19}$$

This equality splits into one part for a right wave and another for a left wave:

$$\nabla\eta^1 = e^{i(\lambda\varphi - Ex^0+\delta)}\Big[\Omega\Big(-iE\eta - \sigma_3\partial_r\eta - \frac{1}{r}\sigma_1\partial_\theta\eta - \frac{i\lambda}{r\sin\theta}\sigma_2\eta\Big)\Big],$$
$$\nabla\widehat{\xi}^1 = e^{-i(\lambda\varphi - Ex^0+\delta)}\Big[\Omega\Big(iE\widehat{\xi} - \sigma_3\partial_r\widehat{\xi} - \frac{1}{r}\sigma_1\partial_\theta\widehat{\xi} + \frac{i\lambda}{r\sin\theta}\sigma_2\widehat{\xi}\Big)\Big]. \tag{C.20}$$

For the hydrogen atom we have: [2]

$$qA = qA^0 = -\frac{\alpha}{r}\ ;\quad \alpha := \frac{e^2}{\hbar c} \approx \frac{1}{137}, \tag{C.21}$$

2. For the Schrödinger equation the same potential is used. But the motion is supposed to be around the center of gravity of the hydrogen atom. A simple correction is made, using quantum mechanics for the motion around the center of gravity. This gives a tiny correction between energy levels in the hydrogen case and in the case of deuterium. No such correction is made with the relativistic equation, where the center of gravity does not have the same properties. We thus suppose, as anyone, that the potential has an exact spherical symmetry.

where α is the fine structure constant. We have:

$$\begin{aligned} qA\widehat{\phi}\sigma_{12} &= -\frac{\alpha}{r}\widehat{\phi}i_3 = -\frac{\alpha}{r}\Omega\widehat{X}e^{(\lambda\varphi-Ex^0+\delta)i_3}i_3 \\ &= \Omega\Big(-\frac{\alpha}{r}\widehat{X}i_3\Big)e^{(\lambda\varphi-Ex^0+\delta)i_3}. \end{aligned} \tag{C.22}$$

Also the (C.7) system becomes:

$$-E\widehat{X}i_3-\sigma_3\partial_r\widehat{X}-\frac{1}{r}\sigma_1\partial_\theta\widehat{X}-\frac{\lambda}{r\sin\theta}\sigma_2\widehat{X}i_3-\frac{\alpha}{r}\widehat{X}i_3+e^{-i\beta}X\mathbf{m}i_3=0, \tag{C.23}$$

and this means:

$$\Big(E+\frac{\alpha}{r}\Big)\widehat{X}i_3+\sigma_3\partial_r\widehat{X}+\frac{1}{r}\sigma_1\partial_\theta\widehat{X}+\frac{\lambda}{r\sin\theta}\sigma_2\widehat{X}i_3=e^{-i\beta}X\mathbf{m}i_3, \tag{C.24}$$

while the Dirac equation gives:

$$\Big(E+\frac{\alpha}{r}\Big)\widehat{X}i_3+\sigma_3\partial_r\widehat{X}+\frac{1}{r}\sigma_1\partial_\theta\widehat{X}+\frac{\lambda}{r\sin\theta}\sigma_2\widehat{X}i_3=mXi_3. \tag{C.25}$$

Now we let:

$$X=\begin{pmatrix}\mathbf{a} & -\mathbf{b}^* \\ \mathbf{c} & \mathbf{d}^*\end{pmatrix}=\begin{pmatrix}\xi & \widehat{\eta}\end{pmatrix}, \tag{C.26}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are functions with complex values of the real variables r and θ. We get:

$$\widehat{X}=\begin{pmatrix}\mathbf{d} & -\mathbf{c}^* \\ \mathbf{b} & \mathbf{a}^*.\end{pmatrix}=\begin{pmatrix}\eta & \widehat{\xi}\end{pmatrix} \tag{C.27}$$

We thus get:

$$e^{-i\beta}(\mathbf{l}\xi \quad \mathbf{r}\widehat{\eta})i_3=ie^{-i\beta}X\mathbf{m}\sigma_3=ie^{-i\beta}\begin{pmatrix}\mathbf{la} & \mathbf{rb}^* \\ \mathbf{lc} & -\mathbf{rd}^*\end{pmatrix}, \tag{C.28}$$

$$\widehat{X}i_3=\begin{pmatrix}\mathbf{d} & -\mathbf{c}^* \\ \mathbf{b} & \mathbf{a}^*\end{pmatrix}\begin{pmatrix}i & 0 \\ 0 & -i\end{pmatrix}=\begin{pmatrix}i\mathbf{d} & i\mathbf{c}^* \\ i\mathbf{b} & -i\mathbf{a}^*\end{pmatrix}, \tag{C.29}$$

$$\sigma_3\partial_r\widehat{X}=\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\partial_r\mathbf{d} & -\partial_r\mathbf{c}^* \\ \partial_r\mathbf{b} & \partial_r\mathbf{a}^*\end{pmatrix}=\begin{pmatrix}\partial_r\mathbf{d} & -\partial_r\mathbf{c}^* \\ -\partial_r\mathbf{b} & -\partial_r\mathbf{a}^*\end{pmatrix}, \tag{C.30}$$

$$\sigma_1\partial_\theta\widehat{X}=\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\partial_\theta\mathbf{d} & -\partial_\theta\mathbf{c}^* \\ \partial_\theta\mathbf{b} & \partial_\theta\mathbf{a}^*\end{pmatrix}=\begin{pmatrix}\partial_\theta\mathbf{b} & \partial_\theta\mathbf{a}^* \\ \partial_\theta\mathbf{d} & -\partial_\theta\mathbf{c}^*\end{pmatrix}, \tag{C.31}$$

$$\sigma_2\widehat{X}i_3=i_2\widehat{X}\sigma_3=\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{d} & -\mathbf{c}^* \\ \mathbf{b} & \mathbf{a}^*\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}=\begin{pmatrix}\mathbf{b} & -\mathbf{a}^* \\ -\mathbf{d} & -\mathbf{c}^*\end{pmatrix}. \tag{C.32}$$

Therefore the improved equation gives:

$$\begin{aligned} &\Big(E+\frac{\alpha}{r}\Big)\begin{pmatrix}i\mathbf{d} & i\mathbf{c}^* \\ i\mathbf{b} & -i\mathbf{a}^*\end{pmatrix}+\begin{pmatrix}\partial_r\mathbf{d} & -\partial_r\mathbf{c}^* \\ -\partial_r\mathbf{b} & -\partial_r\mathbf{a}^*\end{pmatrix}+\frac{1}{r}\begin{pmatrix}\partial_\theta\mathbf{b} & \partial_\theta\mathbf{a}^* \\ \partial_\theta\mathbf{d} & -\partial_\theta\mathbf{c}^*\end{pmatrix} \\ &+\frac{\lambda}{r\sin\theta}\begin{pmatrix}\mathbf{b} & -\mathbf{a}^* \\ -\mathbf{d} & -\mathbf{c}^*\end{pmatrix}=ie^{-i\beta}\begin{pmatrix}\mathbf{la} & \mathbf{rb}^* \\ \mathbf{lc} & -\mathbf{rd}^*\end{pmatrix}. \end{aligned} \tag{C.33}$$

Conjugating the equations containing the $*$ we get the system:

$$\begin{aligned}
i\Big(E+\frac{\alpha}{r}\Big)\mathbf{d}+\partial_r\mathbf{d}+\frac{1}{r}\Big(\partial_\theta+\frac{\lambda}{\sin\theta}\Big)\mathbf{b}&=i\mathbf{l}e^{-i\beta}\mathbf{a},\\
-i\Big(E+\frac{\alpha}{r}\Big)\mathbf{c}-\partial_r\mathbf{c}+\frac{1}{r}\Big(\partial_\theta-\frac{\lambda}{\sin\theta}\Big)\mathbf{a}&=-i\mathbf{r}e^{i\beta}\mathbf{b},\\
i\Big(E+\frac{\alpha}{r}\Big)\mathbf{b}-\partial_r\mathbf{b}+\frac{1}{r}\Big(\partial_\theta-\frac{\lambda}{\sin\theta}\Big)\mathbf{d}&=i\mathbf{l}e^{-i\beta}\mathbf{c},\\
-i\Big(E+\frac{\alpha}{r}\Big)\mathbf{a}+\partial_r\mathbf{a}+\frac{1}{r}\Big(\partial_\theta+\frac{\lambda}{\sin\theta}\Big)\mathbf{c}&=-i\mathbf{r}e^{i\beta}\mathbf{d}.
\end{aligned}\tag{C.34}$$

Moreover we have:

$$\rho e^{i\beta}=\det(\phi)=\frac{\det(X)}{r^2\sin\theta}=\frac{\mathbf{ad}^*+\mathbf{cb}^*}{r^2\sin\theta}\tag{C.35}$$

We thus obtain:

$$e^{i\beta}=\frac{\mathbf{ad}^*+\mathbf{cb}^*}{|\mathbf{ad}^*+\mathbf{cb}^*|}.\tag{C.36}$$

In the equations (C.34), only two angular operators are present, and so we let:

$$\mathbf{a}:=AU\ ;\quad \mathbf{b}:=BV\ ;\quad \mathbf{c}:=CV\ ;\quad \mathbf{d}:=DU,\tag{C.37}$$

where A, B, C et D are functions of r while U and V are functions of θ. The (C.34) system becomes:

$$\begin{aligned}
i\Big(E+\frac{\alpha}{r}\Big)DU+D'U+\frac{1}{r}\Big(V'+\frac{\lambda}{\sin\theta}V\Big)B&=i\mathbf{l}e^{-i\beta}AU,\\
-i\Big(E+\frac{\alpha}{r}\Big)CV-C'V+\frac{1}{r}\Big(U'-\frac{\lambda}{\sin\theta}U\Big)A&=-i\mathbf{r}e^{i\beta}BV,\\
i\Big(E+\frac{\alpha}{r}\Big)BV-B'V+\frac{1}{r}\Big(U'-\frac{\lambda}{\sin\theta}U\Big)D&=i\mathbf{l}e^{-i\beta}CV,\\
-i\Big(E+\frac{\alpha}{r}\Big)AU+A'U+\frac{1}{r}\Big(V'+\frac{\lambda}{\sin\theta}V\Big)C&=-i\mathbf{r}e^{i\beta}DU.
\end{aligned}\tag{C.38}$$

Then if a κ constant exists such as:

$$U'-\frac{\lambda}{\sin\theta}U=-\kappa V\ ;\quad V'+\frac{\lambda}{\sin\theta}V=\kappa U,\tag{C.39}$$

the (C.38) system becomes:

$$\begin{aligned}
i\Big(E+\frac{\alpha}{r}\Big)D+D'+\frac{\kappa}{r}B&=i\mathbf{l}e^{-i\beta}A,\\
-i\Big(E+\frac{\alpha}{r}\Big)C-C'-\frac{\kappa}{r}A&=-i\mathbf{r}e^{i\beta}B,\\
i\Big(E+\frac{\alpha}{r}\Big)B-B'-\frac{\kappa}{r}D&=i\mathbf{l}e^{-i\beta}C,\\
-i\Big(E+\frac{\alpha}{r}\Big)A+A'+\frac{\kappa}{r}C&=-i\mathbf{r}e^{i\beta}D.
\end{aligned}\tag{C.40}$$

To get the system equivalent to the Dirac equation, it is enough to suppress the angle β and to assume the equality $\mathbf{l} = \mathbf{r} = m$. This does not change the angular system (C.39), while in the place of (C.40) we obtain the following system:

$$\begin{aligned} i\Big(E+\frac{\alpha}{r}\Big)D + D' + \frac{\kappa}{r}B &= imA, \\ -i\Big(E+\frac{\alpha}{r}\Big)C - C' - \frac{\kappa}{r}A &= -imB, \\ i\Big(E+\frac{\alpha}{r}\Big)B - B' - \frac{\kappa}{r}D &= imC, \\ -i\Big(E+\frac{\alpha}{r}\Big)A + A' + \frac{\kappa}{r}C &= -imD. \end{aligned} \tag{C.41}$$

C.2 Kinetic momentum operators

We established in [15] the form that the operators of kinetic momentum have in space-time. With the Pauli algebra we have (a detailed calculation is in [23] A.3):

$$\begin{aligned} \mathrm{J}_1\phi &= \Big(d_1+\frac{1}{2}\sigma_{23}\Big)\phi\sigma_{21}\ ;\quad d_1 = x^2\partial_3 - x^3\partial_2 = -\sin\varphi\ \partial_\theta - \frac{\cos\varphi}{\tan\theta}\partial_\varphi, \\ J_2\phi &= \Big(d_2+\frac{1}{2}\sigma_{31}\Big)\phi\sigma_{21}\ ;\quad d_2 = x^3\partial_1 - x^1\partial_3 = \cos\varphi\ \partial_\theta - \frac{\sin\varphi}{\tan\theta}\partial_\varphi, \\ J_3\phi &= \Big(d_3+\frac{1}{2}\sigma_{12}\Big)\phi\sigma_{21}\ ;\quad d_3 = x^1\partial_2 - x^2\partial_1 = \partial_\varphi. \end{aligned} \tag{C.42}$$

We indeed also have:

$$J^2 = J_1^2 + J_2^2 + J_3^2. \tag{C.43}$$

From (C.42) we obtain:

$$J_3\phi = \lambda\phi \Longleftrightarrow \phi = \phi(x^0,\ r,\ \theta)e^{\lambda\varphi i_3}. \tag{C.44}$$

Hence the ϕ wave satisfying (C.8) is a proper vector of J_3 and λ is the magnetic quantum number. And for a wave ϕ satisfying (C.8), we have:

$$J^2\phi = j(j+1)\phi, \tag{C.45}$$

if and only if:

$$\partial^2_{\theta\theta}X + \Big[\Big(j+\frac{1}{2}\Big)^2 - \frac{\lambda^2}{\sin^2\theta}\Big]X - \lambda\frac{\cos\theta}{\sin^2\theta}\sigma_{12}X\sigma_{12} = 0. \tag{C.46}$$

At the second order (C.39) implies:

$$0 = U'' + \Big(\kappa^2 - \frac{\lambda^2}{\sin^2\theta}\Big)U + \lambda\frac{\cos\theta}{\sin^2\theta}U \tag{C.47}$$

$$0 = V'' + \Big(\kappa^2 - \frac{\lambda^2}{\sin^2\theta}\Big)V - \lambda\frac{\cos\theta}{\sin^2\theta}V \tag{C.48}$$

$$0 = \partial^2_{\theta\theta}X + \Big(\kappa^2 - \frac{\lambda^2}{\sin^2\theta}\Big)X - \lambda\frac{\cos\theta}{\sin^2\theta}\sigma_{12}X\sigma_{12}, \tag{C.49}$$

and therefore ϕ is a proper vector of J^2, with the proper value $j(j+1)$, if and only if:

$$\kappa^2 = \Big(j + \frac{1}{2}\Big)^2 \ ; \quad |\kappa| = j + \frac{1}{2} \ ; \quad j = |\kappa| - \frac{1}{2}. \tag{C.50}$$

With the definition of S in (C.3) and with (C.8) we can see that the change of φ into $\varphi + 2\pi$ conserves the same value for the wave if and only if λ has a half-integer value. Only in this case is the wave correctly defined. The general results for the angular momentum operators then imply:

$$j = \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \cdots \ ; \quad \kappa = \pm 1,\ \pm 2,\ \pm 3,\ \cdots \ ; \quad \lambda = -j,\ -j+1,\ \cdots\ j-1,\ j. \tag{C.51}$$

To solve the angular system we let, if $\lambda > 0$ and with $C = C(\theta)$:

$$\begin{aligned} U &= \sin^\lambda \theta \Big[\sin\Big(\frac{\theta}{2}\Big) C' - \Big(\kappa + \frac{1}{2} - \lambda\Big) \cos\Big(\frac{\theta}{2}\Big) C \Big], \\ V &= \sin^\lambda \theta \Big[\cos\Big(\frac{\theta}{2}\Big) C' + \Big(\kappa + \frac{1}{2} - \lambda\Big) \sin\Big(\frac{\theta}{2}\Big) C \Big]. \end{aligned} \tag{C.52}$$

While if $\lambda < 0$ we let:

$$\begin{aligned} U &= \sin^{-\lambda} \theta \Big[\cos\Big(\frac{\theta}{2}\Big) C' + \Big(\kappa + \frac{1}{2} + \lambda\Big) \sin\Big(\frac{\theta}{2}\Big) C \Big], \\ V &= \sin^{-\lambda} \theta \Big[- \sin\Big(\frac{\theta}{2}\Big) C' + \Big(\kappa + \frac{1}{2} + \lambda\Big) \cos\Big(\frac{\theta}{2}\Big) C \Big]. \end{aligned} \tag{C.53}$$

The angular system (C.39) is therefore equivalent [12] to the differential equation:

$$0 = C'' + \frac{2|\lambda|}{\tan\theta} C' + \Big[\Big(\kappa + \frac{1}{2}\Big)^2 - \lambda^2 \Big] C. \tag{C.54}$$

The change of variable:

$$z = \cos\theta \ ; \quad f(z) = C[\theta(z)], \tag{C.55}$$

then gives the differential equation of the Gegenbauer polynomials [3]:

$$0 = f''(z) - \frac{1 + 2|\lambda|}{1 - z^2} z f'(z) + \frac{(\kappa + \frac{1}{2})^2 - \lambda^2}{1 - z^2} f(z). \tag{C.56}$$

And we get, as the only integrable function:

$$\frac{C(\theta)}{C(0)} = \sum_{n=0}^{\infty} \frac{(|\lambda| - \kappa - \frac{1}{2})_n (|\lambda| + \kappa + \frac{1}{2})_n}{(\frac{1}{2} + |\lambda|)_n n!} \sin^{2n}\Big(\frac{\theta}{2}\Big), \tag{C.57}$$

3. When we solve the Dirac equation with Darwin's method, meaning with the *ad hoc* operators, we obtain some Legendre polynomials and spherical harmonics. Here, working with ϕ, which is equivalent to employing the Weyl spinors ξ and η, we obtain the Gegenbauer polynomials, and it is the degree of these polynomials that gives the needed quantum number.

with:

$$(a)_0 = 1 \;\; (a)_1 = a, \qquad (a)_n = a(a+1)\dots(a+n-1). \tag{C.58}$$

The $C(0)$ term is a factor of U and V, its argument may be absorbed by the δ of (C.7), and its modulus may be transferred to the radial functions. We can then let $C(0) = 1$, which gives:

$$C(\theta) = \sum_{n=0}^{\infty} \frac{(|\lambda| - \kappa - \frac{1}{2})_n (|\lambda| + \kappa + \frac{1}{2})_n}{(\frac{1}{2} + |\lambda|)_n n!} \sin^{2n}\Big(\frac{\theta}{2}\Big). \tag{C.59}$$

Since we have the (C.50) conditions on λ and κ, an integer n always exists such that:

$$|\lambda| + n = |\kappa + \frac{1}{2}|, \tag{C.60}$$

and this constrains the (C.59) series to be a finite sum, thus U and V are integrable. And since U and V have real values, we have:

$$e^{i\beta} = \frac{AD^*U^2 + CB^*V^2}{|AD^*U^2 + CB^*V^2|}. \tag{C.61}$$

C.3 Resolution of the radial system

We change the radial variable as follows:

$$x = mr \;;\quad \epsilon = \frac{E}{m} \;;\quad a(x) = A(r) = A\Big(\frac{x}{m}\Big), \tag{C.62}$$
$$b(x) = B(r) \;;\quad c(x) = C(r) \;;\quad d(x) = D(r).$$

The (C.41) system becomes:

$$0 = -\Big(\epsilon + \frac{\alpha}{x}\Big)d + id' + i\frac{\kappa}{x}b + e^{-i\beta}\frac{\mathbf{l}}{m}a, \tag{C.63}$$
$$0 = -\Big(\epsilon + \frac{\alpha}{x}\Big)c + ic' + i\frac{\kappa}{x}a + e^{i\beta}\frac{\mathbf{r}}{m}b, \tag{C.64}$$
$$0 = -\Big(\epsilon + \frac{\alpha}{x}\Big)b - ib' - i\frac{\kappa}{x}d + e^{-i\beta}\frac{\mathbf{l}}{m}c, \tag{C.65}$$
$$0 = -\Big(\epsilon + \frac{\alpha}{x}\Big)a - ia' - i\frac{\kappa}{x}c + e^{i\beta}\frac{\mathbf{r}}{m}d. \tag{C.66}$$

To obtain a probability current we use radial functions, as in the angular system, and for the same reason: they are polynomial functions and not infinite series. We thus let:

$$\begin{aligned} a &:= e^{-\Lambda x}x^s(a_0 + a_1 x + \dots + a_n x^n), \\ b &:= e^{-\Lambda x}x^s(b_0 + b_1 x + \dots + b_n x^n), \\ c &:= e^{-\Lambda x}x^s(c_0 + c_1 x + \dots + c_n x^n), \\ d &:= e^{-\Lambda x}x^s(d_0 + d_1 x + \dots + d_n x^n). \end{aligned} \tag{C.67}$$

Equation (C.66) is equivalent to:

$$\begin{aligned}0 = &-\epsilon(\quad + a_0 x + \cdots + a_{n-1}x^n + a_n x^{n+1})\\ &-\alpha(a_0 + a_1 x + \cdots + a_n x^n)\\ &+i\Lambda(\quad + a_0 x + \cdots + a_{n-1}x^n + a_n x^{n+1})\\ &-is(a_0 + a_1 x + \cdots + a_n x^n)\\ &-i(\quad + a_1 x + \cdots + n a_n x^n)\\ &-i\kappa(c_0 + c_1 x + \cdots + c_n x^n)\\ &+\frac{\mathbf{r}}{m}e^{i\beta}(\quad + d_0 x + \cdots + d_{n-1}x^n + d_n x^{n+1}).\end{aligned} \tag{C.68}$$

Similarly equation (C.65) is equivalent to:

$$\begin{aligned}0 = &-\epsilon(\quad + b_0 x + \cdots + b_{n-1}x^n + b_n x^{n+1})\\ &-\alpha(b_0 + b_1 x + \cdots + b_n x^n)\\ &+i\Lambda(\quad + b_0 x + \cdots + b_{n-1}x^n + b_n x^{n+1})\\ &-is(b_0 + b_1 x + \cdots + b_n x^n)\\ &-i(\quad + b_1 x + \cdots + n b_n x^n)\\ &-i\kappa(d_0 + d_1 x + \cdots + d_n x^n)\\ &+\frac{\mathbf{l}}{m}e^{-i\beta}(\quad + c_0 x + \cdots + c_{n-1}x^n + c_n x^{n+1}).\end{aligned} \tag{C.69}$$

Then equation (C.64) is equivalent to:

$$\begin{aligned}0 = &-\epsilon(\quad + c_0 x + \cdots + c_{n-1}x^n + c_n x^{n+1})\\ &-\alpha(c_0 + c_1 x + \cdots + c_n x^n)\\ &-i\Lambda(\quad + c_0 x + \cdots + c_{n-1}x^n + c_n x^{n+1})\\ &+is(c_0 + c_1 x + \cdots + c_n x^n)\\ &+i(\quad + c_1 x + \cdots + n c_n x^n)\\ &+i\kappa(a_0 + a_1 x + \cdots + a_n x^n)\\ &+\frac{\mathbf{r}}{m}e^{i\beta}(\quad + b_0 x + \cdots + b_{n-1}x^n + b_n x^{n+1}).\end{aligned} \tag{C.70}$$

And equation (C.63) is equivalent to:

$$\begin{aligned}0 = &-\epsilon(\quad + d_0 x + \cdots + d_{n-1}x^n + d_n x^{n+1})\\ &-\alpha(d_0 + d_1 x + \cdots + d_n x^n)\\ &-i\Lambda(\quad + d_0 x + \cdots + d_{n-1}x^n + d_n x^{n+1})\\ &+is(d_0 + d_1 x + \cdots + d_n x^n)\\ &+i(\quad + d_1 x + \cdots + n d_n x^n)\\ &+i\kappa(b_0 + b_1 x + \cdots + b_n x^n)\\ &+\frac{\mathbf{l}}{m}e^{-i\beta}(\quad + a_0 x + \cdots + a_{n-1}x^n + a_n x^{n+1}).\end{aligned} \tag{C.71}$$

We thus obtain three kinds of systems: with index 0, with index between 0 and n, and with index n. With the null index the system depends only on α, κ and s:

$$\begin{aligned} 0 &= (-\alpha - is)a_0 - i\kappa c_0 \ ; \ \ 0 = (-\alpha - is)b_0 - i\kappa d_0, \\ 0 &= i\kappa a_0 + (-\alpha + is)c_0 \ ; \ \ 0 = i\kappa b_0 + (-\alpha + is)d_0. \end{aligned} \tag{C.72}$$

This system is exactly the same as with the linear Dirac equation. It is constituted by two subsystems. We obtain a nonzero solution only if the determinant of each subsystem is null, and thus only if s is such that:

$$0 = (-\alpha - is)(-\alpha + is) - \kappa^2; \ \kappa^2 = s^2 + \alpha^2; \ s = \sqrt{\kappa^2 - \alpha^2}. \tag{C.73}$$

The condition $\kappa \neq 0$ comes from here. System (C.72) is then reduced to:

$$c_0 = \frac{i\alpha - s}{\kappa} a_0; \ d_0 = \frac{i\alpha - s}{\kappa} b_0. \tag{C.74}$$

With the n index we have the following system :

$$\begin{aligned} 0 &= (-\epsilon + i\Lambda)a_n + \frac{\mathbf{r}}{m} e^{i\beta} d_n; \ 0 = (-\epsilon + i\Lambda)b_n + \frac{\mathbf{l}}{m} e^{-i\beta} c_n; \\ 0 &= (-\epsilon - i\Lambda)c_n + \frac{\mathbf{r}}{m} e^{i\beta} b_n; \ 0 = (-\epsilon - i\Lambda)d_n + \frac{\mathbf{l}}{m} e^{-i\beta} a_n. \end{aligned} \tag{C.75}$$

This system forms two similar subsystems, with same determinant D. A nonzero solution exists only if the determinant is null, which gives:

$$\begin{aligned} 0 = D &= \begin{vmatrix} -\epsilon + i\Lambda & \frac{\mathbf{r}}{m} e^{i\beta} \\ \frac{\mathbf{l}}{m} e^{-i\beta} & -\epsilon - i\Lambda \end{vmatrix} = \epsilon^2 + \Lambda^2 - \frac{\mathbf{lr}}{m^2}, \\ 0 = D &\Leftrightarrow \frac{\mathbf{lr}}{m^2} = \epsilon^2 + \Lambda^2, \end{aligned} \tag{C.76}$$

To solve the linear Dirac equation we used to let $\epsilon^2 + \Lambda^2 = 1$. This should be the case if there would be equality between $\mathbf{l}$ and $\mathbf{r}$. System (C.75) is equivalent to:

$$d_n = \frac{\mathbf{l}}{m} e^{-i\beta}(\epsilon - i\Lambda)a_n, \tag{C.77}$$

$$c_n = \frac{\mathbf{r}}{m} e^{i\beta}(\epsilon - i\Lambda)b_n. \tag{C.78}$$

With the $n = 0$ case, that is the case of radial polynomials reduced to constant terms, the radial system is reduced to (C.74), (C.77) and (C.78), and we thus have:

$$d_0 = \frac{\mathbf{l}}{m} e^{-i\beta}(\epsilon - i\Lambda)a_0 = \frac{i\alpha - s}{\kappa} b_0, \tag{C.79}$$

$$c_0 = \frac{\mathbf{r}}{m} e^{i\beta}(\epsilon - i\Lambda)b_0 = \frac{i\alpha - s}{\kappa} a_0. \tag{C.80}$$

This gives the following system:

$$\frac{\mathbf{l}}{m}e^{-i\beta}(\epsilon - i\Lambda)a_0 - \frac{i\alpha - s}{\kappa}b_0 = 0, \tag{C.81}$$
$$-\frac{i\alpha - s}{\kappa}a_0 + \frac{\mathbf{r}}{m}e^{i\beta}(\epsilon - i\Lambda)b_0 = 0.$$

This system has a nonzero solution only if its determinant is null, which gives:

$$0 = \frac{\mathbf{lr}}{m^2}(\epsilon - i\Lambda)^2 - \frac{(s - i\alpha)^2}{\kappa^2}, \tag{C.82}$$
$$\frac{\sqrt{\mathbf{lr}}}{m}(\epsilon - i\Lambda) = \pm\frac{s - i\alpha}{|\kappa|}. \tag{C.83}$$

The separation of this equation into real and imaginary parts allows us only the + sign, because s, α, ϵ and Λ are positive. That is how the Dirac theory obtains the true number $2\mathbf{n}^2$ for the energy levels with principal quantum number $\mathbf{n} := |\kappa| + n$ [54] (explained in 1.5.7). That gives:

$$\Lambda = \nu\frac{\alpha}{\kappa};\ \epsilon = \nu\frac{s}{\kappa};\ \alpha\epsilon = s\Lambda, \tag{C.84}$$
$$\nu^2 = \epsilon^2 + \Lambda^2 = \epsilon^2 + \frac{\epsilon^2\alpha^2}{s^2} = \frac{\epsilon^2(s^2+\alpha^2)}{s^2} = \frac{\epsilon^2\kappa^2}{s^2}, \tag{C.85}$$
$$\epsilon = \frac{\nu s}{\kappa} = \nu\sqrt{1 - \frac{\alpha^2}{\kappa^2}} = \frac{\nu}{\sqrt{1 + \frac{\alpha^2}{(n+s)^2}}}. \tag{C.86}$$

We hence obtain Sommerfeld's formula (C.86) with $n = 0$ and ν instead of 1 in the numerator. To account for the Lamb effect, which increases the energy of $1\mathbf{s}1/2$ states by 8.2 GHz, it is enough to take, for $\kappa = 1$:

$$\nu = 1 + 6.615 \times 10^{-11}. \tag{C.87}$$

With $n > 0$ the system with index between 0 and n is:

$$0 = (-\epsilon + i\Lambda)a_{n-1} - [\alpha + i(s+n)]a_n - i\kappa c_n + \frac{\mathbf{r}}{m}e^{i\beta}d_{n-1}, \tag{C.88}$$
$$0 = (-\epsilon + i\Lambda)b_{n-1} - [\alpha + i(s+n)]b_n - i\kappa d_n + \frac{\mathbf{l}}{m}e^{-i\beta}c_{n-1}, \tag{C.89}$$
$$0 = (-\epsilon - i\Lambda)c_{n-1} + [-\alpha + i(s+n)]c_n + i\kappa a_n + \frac{\mathbf{r}}{m}e^{i\beta}b_{n-1}, \tag{C.90}$$
$$0 = (-\epsilon - i\Lambda)d_{n-1} + [-\alpha + i(s+n)]d_n + i\kappa b_n + \frac{\mathbf{l}}{m}e^{-i\beta}a_{n-1},. \tag{C.91}$$

Hence we multiply (C.88) by $\frac{1}{m}e^{-i\beta}$, likewise (C.91) by $\epsilon - i\Lambda$, and we add the products. This suppresses terms with index $n-1$, and yields:

$$
\begin{aligned}
0 = & -[\alpha + i(s+n)]\frac{\mathbf{l}}{m}e^{-i\beta}a_n + i\kappa(\epsilon - i\Lambda)b_n \\
& - i\kappa\frac{\mathbf{l}}{m}e^{-i\beta}c_n + [-\alpha + i(s+n)](\epsilon - i\Lambda)d_n
\end{aligned} \tag{C.92}
$$

Using (C.77) and (C.78) this gives:

$$
\begin{aligned}
0 = & \Big[-\frac{[\alpha + i(s+n)]}{\epsilon - i\Lambda} + [-\alpha + i(s+n)](\epsilon - i\Lambda)\Big]d_n \\
& + i\kappa\Big[\frac{m}{\mathbf{r}} - \frac{\mathbf{l}}{m}\Big]e^{-i\beta}c_n
\end{aligned} \tag{C.93}
$$

This reveals a relation between d_n and c_n. We may obtain another relation by multiplying (C.89) by $\frac{\mathbf{r}}{m}e^{i\beta}$, likewise (C.90) by $\epsilon - i\Lambda$, and then adding the products, which gives:

$$
\begin{aligned}
0 = & i\kappa(\epsilon - i\Lambda)a_n - [\alpha + i(s+n)]\frac{\mathbf{r}}{m}e^{i\beta}b_n \\
& + (\epsilon - i\Lambda)[-\alpha + i(s+n)]c_n - i\kappa\frac{\mathbf{r}}{m}e^{i\beta}d_n.
\end{aligned} \tag{C.94}
$$

Using again (C.77) and (C.78) we obtain another equation with only d_n and c_n:

$$
0 = i\kappa\Big(\frac{m}{\mathbf{l}} - \frac{\mathbf{r}}{m}\Big)e^{i\beta}d_n + 2[\Lambda(s+n) - \alpha\epsilon]c_n. \tag{C.95}
$$

We thus consider the following system:

$$
\begin{aligned}
0 = & \Big[-\frac{[\alpha + i(s+n)]}{\epsilon - i\Lambda} + [-\alpha + i(s+n)](\epsilon - i\Lambda)\Big]d_n \\
& + i\kappa\Big[\frac{m}{\mathbf{r}} - \frac{\mathbf{l}}{m}\Big]e^{-i\beta}c_n, \\
0 = & i\kappa\Big(\frac{m}{\mathbf{l}} - \frac{\mathbf{r}}{m}\Big)e^{i\beta}d_n + 2[\Lambda(s+n) - \alpha\epsilon]c_n.
\end{aligned} \tag{C.96}
$$

The necessity of normalizing the wave implies that the series we used must be polynomial functions. Hence an integer n must exist such that $a_n \dots d_n$ are all zero. The determinant of the (C.96) system must thus be null:

$$
\begin{aligned}
0 = & \frac{\mathbf{lr}}{m^2}\Big[-[\alpha + i(s+n)] + [-\alpha + i(s+n)](\epsilon - i\Lambda)^2\Big]^2 \\
& + \kappa^2(\epsilon - i\Lambda)^2(1 - \epsilon^2 - \Lambda^2)^2.
\end{aligned} \tag{C.97}
$$

Hence, dividing by $\nu^2 = \frac{\mathbf{lr}}{m^2} = \epsilon^2 + \Lambda^2$ we obtain:

$$
\begin{aligned}
0 = & \Big[-[\alpha + i(s+n)] + [-\alpha + i(s+n)](\epsilon - i\Lambda)^2\Big]^2 \\
& + \kappa^2(\epsilon - i\Lambda)^2\Big(\nu - \frac{1}{\nu}\Big)^2.
\end{aligned} \tag{C.98}
$$

To simplify the calculation, we let:

$$s + n =: \alpha S; \quad \tau := \frac{\kappa}{2\alpha}\Big(\nu - \frac{1}{\nu}\Big), \tag{C.99}$$

which gives:

$$0 = \Big[-(1+iS) + (-1+iS)(\epsilon - i\Lambda)^2 \Big]^2 - \Big[\frac{\kappa}{\alpha}(\Lambda + i\epsilon)\Big(\nu - \frac{1}{\nu}\Big)\Big]^2. \tag{C.100}$$

This equation is simplified if a unique mass term exists, which means $\nu = 1$. We then have:

$$0 = -(1+iS) + (-1+iS)(\epsilon - i\Lambda)^2. \tag{C.101}$$

Separating the real and imaginary parts, we obtain:

$$0 = -1 - \epsilon^2 + \Lambda^2 + 2S\epsilon\Lambda, \tag{C.102}$$

$$0 = -S + S(\epsilon^2 - \Lambda^2) + 2\epsilon\Lambda. \tag{C.103}$$

This last equation gives:

$$\begin{aligned} \epsilon\Lambda &= \frac{S}{1+S^2};\ 0 = -1 - \epsilon^2 + (1-\epsilon^2) + 2S\frac{S}{1+S^2}, \\ \epsilon^2 &= \frac{S^2}{1+S^2} = \frac{(s+n)^2}{\alpha^2 + (s+n)^2};\ \epsilon = \frac{1}{\sqrt{1 + \dfrac{\alpha^2}{(s+n)^2}}}. \end{aligned} \tag{C.104}$$

This is Sommerfeld's formula obtained with the Dirac equation. We know that this formula cannot account for the Lamb effect.

If we don't suppose $\nu = 1$, (C.100) may be read as:

$$\begin{aligned} (A+iB)^2 = (C+iD)^2;\ A = -1 - \epsilon^2 + \Lambda^2 + 2S\epsilon\Lambda;\ C = 2\Lambda\tau, \\ B = -S + 2\epsilon\Lambda + S(\epsilon^2 - \Lambda^2);\ D = 2\epsilon\tau. \end{aligned} \tag{C.105}$$

There are two possibilities: either $A = C$ and $B = D$, or $A = -C$ and $B = -D$. The first possibility gives us:

$$-1 - \epsilon^2 + \Lambda^2 + 2S\epsilon\Lambda = 2\Lambda\tau, \tag{C.106}$$

$$-S + 2\epsilon\Lambda + S(\epsilon^2 - \Lambda^2) = 2\epsilon\tau. \tag{C.107}$$

We use with (C.106), the relation $\Lambda^2 = \nu^2 - \epsilon^2$, which gives:

$$2(S\epsilon - \tau)\Lambda = 1 + \epsilon^2 - (\nu^2 - \epsilon^2). \tag{C.108}$$

Thus we have:

$$\Lambda = \frac{2\epsilon^2 + 1 - \nu^2}{2(S\epsilon - \tau)}. \tag{C.109}$$

Next we use (C.107):

$$2\epsilon\Lambda = S(1+\nu^2-2\epsilon^2)+2\epsilon\tau, \tag{C.110}$$

$$2\epsilon\frac{2\epsilon^2+1-\nu^2}{2(S\epsilon-\tau)} = S(1+\nu^2-2\epsilon^2)+2\epsilon\tau, \tag{C.111}$$

$$2\epsilon^3+(1-\nu^2)\epsilon = (S\epsilon-\tau)[S(1+\nu^2-2\epsilon^2)+2\epsilon\tau]. \tag{C.112}$$

This gives the cubic equation:

$$0=\epsilon^3-\frac{2S\tau}{1+S^2}\epsilon^2+\frac{1-\nu^2+2\tau^2-S^2(1+\nu^2)}{2(1+S^2)}\epsilon+\frac{S\tau(1+\nu^2)}{2(1+S^2)}. \tag{C.113}$$

To simplify the calculation, we use:

$$0=\epsilon^3-a\epsilon^2-b\epsilon+c, \tag{C.114}$$

$$a := \frac{2S\tau}{1+S^2} = \frac{\kappa(s+n)(\nu-1/\nu)}{2[(s+n)^2+\alpha^2)]}, \tag{C.115}$$

$$b := \frac{-1+\nu^2-2\tau^2+S^2(1+\nu^2)}{2(1+S^2)}, \tag{C.116}$$

$$c := \frac{S\tau(1+\nu^2)}{2(1+S^2)} = a\frac{1+\nu^2}{4}. \tag{C.117}$$

Next we search for a solution such as:

$$\epsilon=\frac{a}{3}+u+v, \tag{C.118}$$

where u and v are two quantities to be determined, and u and v must satisfy:

$$(\epsilon-a/3)^3=(u+v)^3;\ \epsilon^3-a\epsilon^2+\frac{a^2}{3}\epsilon-\frac{a^3}{27}=u^3+3u^2v+3uv^2+v^3,$$

$$\epsilon^3-a\epsilon^2=b\epsilon-c=-\frac{a^2}{3}\epsilon+\frac{a^3}{27}+u^3+v^3+3uv(\epsilon-\frac{a}{3}), \tag{C.119}$$

$$(b+\frac{a^2}{3}-3uv)\epsilon=c+\frac{a^3}{27}+u^3+v^3-auv. \tag{C.120}$$

This equality is satisfied by:

$$uv=\frac{b}{3}+\frac{a^2}{9};\ u^3+v^3=-c+\frac{ab}{3}+\frac{2a^3}{27}. \tag{C.121}$$

This sufficient solution gives:

$$uv=\frac{-1+\nu^2-2\tau^2+S^2(1+\nu^2)}{6(1+S^2)}+\frac{4S^2\tau^2}{9(1+S^2)^2},$$

$$\mathbf{P}:=u^3v^3=\Big[\frac{-1+\nu^2-2\tau^2+S^2(1+\nu^2)}{6(1+S^2)}+\frac{4S^2\tau^2}{9(1+S^2)^2}\Big]^3, \tag{C.122}$$

$$\begin{aligned}\mathbf{S} := u^3 + v^3 = {} & \frac{2S\tau[-1+\nu^2-2\tau^2+S^2(1+\nu^2)]}{6(1+S^2)^2} \\ & + \frac{2(2S\tau)^3}{[3(1+S^2)]^3} - \frac{S\tau(1+\nu^2)}{2(1+S^2)}\end{aligned} \tag{C.123}$$

That allows us to reduce the calculation to the resolution of the following equation:

$$\begin{aligned}0 &= x^2 - \mathbf{S}x + \mathbf{P}, \\ \Delta &= \mathbf{S}^2 - 4\mathbf{P}.\end{aligned} \tag{C.124}$$

The rest depends on the sign of Δ. For instance, for $2\mathbf{s}1/2$ and $2\mathbf{p}1/2$ states, in the first case, we suppose:

$$\nu = 1 + 9.29 \times 10^{-12}, \tag{C.125}$$

which gives;

$$\begin{aligned} n = 1;\ \kappa &= 1;\ s = \sqrt{1-\alpha^2} \approx 0.999973374, \\ \nu - 1/\nu &\approx 1.85798 \times 10^{-11};\ S \approx 274.06834955145, \\ \tau &\approx 1.273051 \times 10^{-9};\ a \approx 9.2899 \times 10^{-12} \\ b &\approx 0.999986686993;\ c \approx 4.64495 \times 10^{-12}, \\ uv &\approx 0.333328895664;\ \mathbf{S} \approx -1.548358 \times 10^{-12} \\ \mathbf{P} &\approx 0.037035557834;\ \Delta \approx -0.14814223134.\end{aligned} \tag{C.126}$$

The quadratic equation thus has two conjugate solutions:

$$u^3 = \frac{\mathbf{S} + i\sqrt{-\mathbf{S}^2+4\mathbf{P}}}{2};\ v^3 = \frac{\mathbf{S} - i\sqrt{\mathbf{S}^2-4\mathbf{P}}}{2} \tag{C.127}$$

$$|u^3| = [\mathbf{S}^2 + (-\mathbf{S}^2+4\mathbf{P})]^{1/2}/2 = \sqrt{\mathbf{P}}. \tag{C.128}$$

We thus let:

$$u^3 := \sqrt{\mathbf{P}}e^{3i\delta}, \tag{C.129}$$

that gives:

$$\tan(3\delta) = \frac{\sqrt{-\mathbf{S}^2+4\mathbf{P}}}{\mathbf{S}};\ 3\delta \approx -1.570796326791, \tag{C.130}$$

$$\begin{aligned}\delta &= \frac{1}{3}\tan^{-1}\left(\sqrt{\frac{4\mathbf{P}}{\mathbf{S}^2}-1}\right)\mathrm{mod}\frac{2\pi}{3} \approx -0.523598775597, \\ u &= \mathbf{P}^{\frac{1}{6}}e^{i\delta};\ v = \overline{u} = \mathbf{P}^{\frac{1}{6}}e^{-i\delta}, \\ \epsilon &= \frac{a}{3} + u + v = \frac{a}{3} + 2\mathbf{P}^{\frac{1}{6}}\cos(\delta) \approx 0.9999933434784.\end{aligned} \tag{C.131}$$

That gives Sommerfeld's formula, including the Lamb shift.

For $2\mathbf{p}1/2$ states, supposing $\nu = 1 + 10^{-14}$, we have:

$$\begin{aligned}
&n = 1;\ \kappa = -1;\ s = \sqrt{1-\alpha^2} \approx 0.999973374,\\
&\nu - 1/\nu \approx 2 \times 10^{-14};\ \tau \approx -1.369264 \times 10^{-12}\\
&a \approx -10^{-14};\ b \approx 0.999986686984,\\
&c \approx -5 \times 10^{-15};\ uv \approx 0.333328895661,\\
&\mathbf{S} \approx 1.666 \times 10^{-15};\ \mathbf{P} \approx 0.037035557833,\\
&\Delta \approx -0.14814223133;\ 3\delta \approx 1.570796326795,\\
&\delta \approx 0.523598775598;\ \epsilon \approx 0.99999334346992.
\end{aligned} \tag{C.132}$$

To obtain (C.100) the second possibility is such that:

$$-1 - \epsilon^2 + \Lambda^2 + 2S\epsilon\Lambda = -2\Lambda\tau, \tag{C.133}$$

$$-S + 2\epsilon\Lambda + S(\epsilon^2 - \Lambda^2) = -2\epsilon\tau. \tag{C.134}$$

We use in (C.133) the relation $\Lambda^2 = \nu^2 - \epsilon^2$, which gives:

$$2(S\epsilon + \tau)\Lambda = 1 + \epsilon^2 - (\nu^2 - \epsilon^2). \tag{C.135}$$

That gives:

$$\Lambda = \frac{2\epsilon^2 + 1 - \nu^2}{2(S\epsilon + \tau)}. \tag{C.136}$$

Then we use (C.134):

$$2\epsilon\Lambda = S(1 + \nu^2 - 2\epsilon^2) - 2\epsilon\tau, \tag{C.137}$$

$$2\epsilon\frac{2\epsilon^2 + 1 - \nu^2}{2(S\epsilon + \tau)} = S(1 + \nu^2 - 2\epsilon^2) - 2\epsilon\tau, \tag{C.138}$$

$$2\epsilon^3 + (1 - \nu^2)\epsilon = (S\epsilon + \tau)[S(1 + \nu^2 - 2\epsilon^2) - 2\epsilon\tau]. \tag{C.139}$$

This leads to the cubic equation:

$$0 = \epsilon^3 + \frac{2S\tau}{1+S^2}\epsilon^2 + \frac{1 - \nu^2 + 2\tau^2 - S^2(1+\nu^2)}{2(1+S^2)}\epsilon - \frac{S\tau(1+\nu^2)}{2(1+S^2)}. \tag{C.140}$$

And this equation is what we obtain from (C.113) by changing the sign of κ and thus τ. We thus obtain the same results.

C.4 Tensor densities

The wave function ϕ is not the only object of Dirac theory. Tensor densities, with or without partial derivatives of ϕ, are also important quantities to investigate: this is the aim of this section. We encountered in Chapter 1 four space-time vectors: $\mathrm{D} = \phi\sigma_\mu\phi^\dagger$, and four other quantities without derivatives of ϕ, which are $\phi\sigma_\mu\overline{\phi}$. Moreover, we obtain two kinds of currents, the $\mathrm{J} = \mathrm{D}_0 = \phi\phi^\dagger$ current, and also the $\underline{\mathbf{J}}$ current defined in (1.203), which is proportional to J only if $\mathbf{l} = \mathbf{r}$. We have:

$$\begin{aligned}
\underline{\mathbf{J}} &= \frac{m}{k\mathbf{l}} L^1 L^{1\dagger} + \frac{m}{k\mathbf{r}} R^1 R^{1\dagger}, && \text{(C.141)}\\
&= \frac{m}{k\mathbf{l}} \Big[\phi\frac{1-\sigma_3}{2}\Big]\Big[\phi\frac{1-\sigma_3}{2}\Big]^\dagger + \frac{m}{k\mathbf{r}} \Big[\phi\frac{1+\sigma_3}{2}\Big]\Big[\phi\frac{1+\sigma_3}{2}\Big]^\dagger\\
&= \frac{m}{k}\phi\Big[\frac{1-\sigma_3}{2\mathbf{l}} + \frac{1+\sigma_3}{2\mathbf{r}}\Big]\phi^\dagger\\
&= \frac{m}{k}\phi\begin{pmatrix}\mathbf{r}^{-1} & 0\\ 0 & \mathbf{l}^{-1}\end{pmatrix}\phi^\dagger = \frac{m}{k}\phi\widehat{\mathbf{m}}^{-1}\phi^\dagger. && \text{(C.142)}
\end{aligned}$$

Using (C.8) and (C.26) we obtain:

$$\begin{aligned}
\underline{\mathbf{J}} &= \frac{m}{k}\Omega X e^{(\lambda\varphi - E\mathrm{x}_0+\delta)i_3}\widehat{\mathbf{m}}^{-1}e^{-(\lambda\varphi - E\mathrm{x}_0+\delta)i_3}X^\dagger\Omega^\dagger\\
&= \frac{m}{k}\Omega X\widehat{\mathbf{m}}^{-1}X^\dagger\Omega^\dagger = \frac{m}{k}\Omega\underline{J}\Omega^\dagger, && \text{(C.143)}\\
\underline{J} &= X\widehat{\mathbf{m}}^{-1}X^\dagger = \begin{pmatrix}\frac{\mathbf{aa}^*}{\mathbf{r}} + \frac{\mathbf{bb}^*}{\mathbf{l}} & \frac{\mathbf{ac}^*}{\mathbf{r}} - \frac{\mathbf{db}^*}{\mathbf{l}}\\ \frac{\mathbf{ca}^*}{\mathbf{r}} - \frac{\mathbf{bd}^*}{\mathbf{l}} & \frac{\mathbf{cc}^*}{\mathbf{r}} + \frac{\mathbf{dd}^*}{\mathbf{l}}\end{pmatrix}. && \text{(C.144)}
\end{aligned}$$

Using (C.37) we obtain, as the probability density:

$$\underline{\mathbf{J}}^0 = \frac{m}{k}\Omega\underline{J}^0\Omega^\dagger = \frac{m}{2kr^2\sin\theta}\Big[\Big(\frac{AA^*}{\mathbf{r}} + \frac{DD^*}{\mathbf{l}}\Big)U^2 + \Big(\frac{CC^*}{\mathbf{r}} + \frac{BB^*}{\mathbf{l}}\Big)V^2\Big]. \tag{C.145}$$

We also have:

$$\begin{aligned}
D_\mu &= \phi\sigma_\mu\phi^\dagger = \Omega X e^{(\lambda\varphi - E\mathrm{x}^0+\delta)i_3}\sigma_\mu e^{-(\lambda\varphi - E\mathrm{x}^0+\delta)i_3}X^\dagger\Omega^\dagger,\\
S_\mu &= \phi\sigma_\mu\overline{\phi} = \Omega X e^{(\lambda\varphi - E\mathrm{x}^0+\delta)i_3}\sigma_\mu e^{-(\lambda\varphi - E\mathrm{x}^0+\delta)i_3}\overline{X}\Omega^\dagger.
\end{aligned}$$

If $\mu = 0$ or $\mu = 3$, we obtain:

$$D_0 = \Omega XX^\dagger\Omega^\dagger;\ D_3 = \Omega X\sigma_3X^\dagger\Omega^\dagger, \tag{C.146}$$

$$S_0 = \Omega X\overline{X}\Omega^\dagger;\ S_3 = \Omega X\sigma_3\overline{X}\Omega^\dagger. \tag{C.147}$$

We also have:

$$\begin{aligned}
XX^\dagger &= \begin{pmatrix}\mathbf{a} & -\mathbf{b}^*\\ \mathbf{c} & \mathbf{d}^*\end{pmatrix}\begin{pmatrix}\mathbf{a}^* & \mathbf{c}^*\\ -\mathbf{b} & \mathbf{d}\end{pmatrix}\\
&= \begin{pmatrix}\mathbf{aa}^* + \mathbf{bb}^* & \mathbf{ac}^* - \mathbf{db}^*\\ \mathbf{ca}^* - \mathbf{bd}^* & \mathbf{cc}^* + \mathbf{dd}^*\end{pmatrix} = \begin{pmatrix}d_0^0 + d_0^3 & d_0^1 - id_0^2\\ d_0^1 + id_0^2 & d_0^0 - d_0^3\end{pmatrix} && \text{(C.148)}\\
X\sigma_3X^\dagger &= \begin{pmatrix}\mathbf{a} & -\mathbf{b}^*\\ \mathbf{c} & \mathbf{d}^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\mathbf{a}^* & \mathbf{c}^*\\ -\mathbf{b} & \mathbf{d}\end{pmatrix}\\
&= \begin{pmatrix}\mathbf{aa}^* - \mathbf{bb}^* & \mathbf{ac}^* + \mathbf{db}^*\\ \mathbf{ca}^* + \mathbf{bd}^* & \mathbf{cc}^* - \mathbf{dd}^*\end{pmatrix}\begin{pmatrix}d_3^0 + d_3^3 & d_3^1 - id_3^2\\ d_3^1 + id_3^2 & d_3^0 - d_3^3\end{pmatrix}. && \text{(C.149)}
\end{aligned}$$

This gives:

$$
\begin{aligned}
2d_0^0 &= \mathbf{aa}^* + \mathbf{bb}^* + \mathbf{cc}^* + \mathbf{dd}^* = (AA^* + DD^*)U^2 + (BB^* + CC^*)V^2,\\
2d_0^3 &= \mathbf{aa}^* + \mathbf{bb}^* - \mathbf{cc}^* - \mathbf{dd}^* = (AA^* - DD^*)U^2 + (BB^* - CC^*)V^2,\\
2d_0^1 &= \mathbf{ca}^* + \mathbf{ac}^* - \mathbf{bd}^* - \mathbf{db}^* = (CA^* + AC^* - BD^* - DB^*)UV,\\
2id_0^2 &= \mathbf{ca}^* - \mathbf{ac}^* - \mathbf{bd}^* + \mathbf{db}^* = (CA^* - AC^* - BD^* + DB^*)UV.
\end{aligned} \tag{C.150}
$$

$$
\begin{aligned}
2d_3^0 &= \mathbf{aa}^* - \mathbf{bb}^* + \mathbf{cc}^* - \mathbf{dd}^* = (AA^* - DD^*)U^2 + (-BB^* + CC^*)V^2,\\
2d_3^3 &= \mathbf{aa}^* - \mathbf{bb}^* - \mathbf{cc}^* + \mathbf{dd}^* = (AA^* + DD^*)U^2 - (BB^* + CC^*)V^2,\\
2d_3^1 &= \mathbf{ca}^* + \mathbf{ac}^* + \mathbf{bd}^* + \mathbf{db}^* = (CA^* + AC^* + BD^* + DB^*)UV,\\
2id_3^2 &= \mathbf{ca}^* - \mathbf{ac}^* + \mathbf{bd}^* - \mathbf{db}^* = (CA^* - AC^* + BD^* - DB^*)UV.
\end{aligned} \tag{C.151}
$$

We also have:

$$
\begin{aligned}
\mathrm{D}_0 = \mathrm{D}_0^\nu \sigma_\nu &= \frac{1}{r\sqrt{\sin\theta}} e^{-\varphi i_3/2} e^{-\theta i_2/2} d_0^\nu \sigma_\mu e^{\theta i_2/2} e^{\varphi i_3/2} \frac{1}{r\sqrt{\sin\theta}}\\
&= \frac{1}{r^2 \sin\theta} e^{-\varphi i_3/2} [d_0^0 + d_0^2\sigma_2 + (d_0^1\sigma_1 + d_0^3\sigma_3)e^{\theta i_2}] e^{\varphi i_3/2} \qquad \text{(C.152)}\\
&= \frac{1}{r^2\sin\theta} \begin{bmatrix} d_0^0 + (d_0^1\cos\theta\cos\varphi - d_0^2\sin\varphi + d_0^3\sin\theta\cos\varphi)\sigma_1 \\ +(d_0^1\cos\theta\sin\varphi + d_0^2\cos\varphi + d_0^3\sin\theta\sin\varphi)\sigma_2 \\ +(-d_0^1\sin\theta + d_0^3\cos\theta)\sigma_3 \end{bmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{D}_3 = \mathrm{D}_3^\nu \sigma_\nu &= \frac{1}{r\sqrt{\sin\theta}} e^{-\varphi i_3/2} e^{-\theta i_2/2} d_3^\nu \sigma_\mu e^{\theta i_2/2} e^{\varphi i_3/2} \frac{1}{r\sqrt{\sin\theta}}\\
&= \frac{1}{r^2 \sin\theta} e^{-\varphi i_3/2} [d_3^0 + d_3^2\sigma_2 + (d_3^1\sigma_1 + d_3^3\sigma_3)e^{\theta i_2}] e^{\varphi i_3/2} \qquad \text{(C.153)}\\
&= \frac{1}{r^2\sin\theta} \begin{bmatrix} d_3^0 + (d_3^1\cos\theta\cos\varphi - d_3^2\sin\varphi + d_3^3\sin\theta\cos\varphi)\sigma_1 \\ +(d_3^1\cos\theta\sin\varphi + d_3^2\cos\varphi + d_3^3\sin\theta\sin\varphi)\sigma_2 \\ +(-d_3^1\sin\theta + d_3^3\cos\theta)\sigma_3 \end{bmatrix}.
\end{aligned}
$$

Similarly with S_μ quantities we have:

$$
\begin{aligned}
X\overline{X} &= \begin{pmatrix} \mathbf{a} & -\mathbf{b}^* \\ \mathbf{c} & \mathbf{d}^* \end{pmatrix} \begin{pmatrix} \mathbf{d}^* & \mathbf{b}^* \\ -\mathbf{c} & \mathbf{a} \end{pmatrix}\\
&= \begin{pmatrix} \mathbf{ad}^* + \mathbf{cb}^* & \mathbf{ab}^* - \mathbf{ab}^* \\ \mathbf{cd}^* - \mathbf{cd}^* & \mathbf{cb}^* + \mathbf{ad}^* \end{pmatrix} = \mathbf{ad}^* + \mathbf{cb}^*, \qquad \text{(C.154)}\\
X\sigma_3\overline{X} &= \begin{pmatrix} \mathbf{a} & -\mathbf{b}^* \\ \mathbf{c} & \mathbf{d}^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{d}^* & \mathbf{b}^* \\ -\mathbf{c} & \mathbf{a} \end{pmatrix}\\
&= \begin{pmatrix} \mathbf{ad}^* - \mathbf{cb}^* & \mathbf{ab}^* + \mathbf{ab}^* \\ \mathbf{cd}^* + \mathbf{cd}^* & \mathbf{cb}^* - \mathbf{ad}^* \end{pmatrix}\\
&= (\mathbf{ad}^* - \mathbf{cb}^*)\sigma_3 + \mathbf{ab}^*(\sigma_1 + i\sigma_2) + \mathbf{cd}^*(\sigma_1 - i\sigma_2). \qquad \text{(C.155)}
\end{aligned}
$$

C.4.1 Low value cases

If $\kappa = 1$ and $\lambda = 1/2$, we obtain:

$$C(\theta) = \sum_{n=0}^{\infty} \frac{(-1)_n (2)_n}{(1)_n n!} \sin^{2n} \frac{\theta}{2} = 1 - 2\sin^2\left(\frac{\theta}{2}\right) = \cos\theta, \tag{C.156}$$

$$C'(\theta) = -\sin\theta;\ U = -\sqrt{\sin\theta}\cos\left(\frac{\theta}{2}\right);\ V = -\sqrt{\sin\theta}\sin\left(\frac{\theta}{2}\right). \tag{C.157}$$

We then have:

$$\begin{aligned} X\overline{X} &= AD^*U^2 + CB^*V^2 = CB^*V^2 \\ &= \frac{\sin\theta}{2}[AD^* + CB^* + \cos\theta(AD^* - CB^*)], \end{aligned} \tag{C.158}$$

$$e^{i\beta} = \frac{X\overline{X}}{|X\overline{X}|} = \frac{AD^* + CB^* + \cos\theta(AD^* - CB^*)}{|AD^* + CB^* + \cos\theta(AD^* - CB^*)|}. \tag{C.159}$$

Thus the Yvon-Takabayasi angle is a function of both the radial parameter r and of the θ angle. Then, if $\kappa = 1$ and $\lambda = -1/2$, we obtain:

$$C(\theta) = \cos\theta;\ C'(\theta) = -\sin\theta, \tag{C.160}$$

$$U = -\sqrt{\sin\theta}\sin\left(\frac{\theta}{2}\right);\ V = \sqrt{\sin\theta}\cos\left(\frac{\theta}{2}\right). \tag{C.161}$$

We thus have:

$$\begin{aligned} X\overline{X} &= AD^*U^2 + CB^*V^2 \\ &= \frac{\sin\theta}{2}[AD^* + CB^* - \cos\theta(AD^* - CB^*)], \end{aligned} \tag{C.162}$$

$$e^{i\beta} = \frac{X\overline{X}}{|X\overline{X}|} = \frac{AD^* + CB^* - \cos\theta(AD^* - CB^*)}{|AD^* + CB^* - \cos\theta(AD^* - CB^*)|}. \tag{C.163}$$

We recall that if $\kappa = 1$ we have $j = |\kappa| - 1/2 = 1/2$, the only possible values for λ are $1/2$ and $-1/2$. Thus as soon as $\kappa = 1$ we only have two kinds of states: first if $\lambda = 1/2$:

$$\begin{aligned} \phi &= \Omega X e^{(\frac{\varphi}{2} - E\mathrm{x}^0)i_3} \\ &= \frac{1}{r\sqrt{\sin\theta}} e^{-\frac{\varphi}{2}i_3} e^{-\frac{\theta}{2}i_2} \begin{pmatrix} AU & -B^*V \\ CV & D^*U \end{pmatrix} e^{(\frac{\varphi}{2} - E\mathrm{x}^0)i_3}. \end{aligned} \tag{C.164}$$

We let:

$$A := r^s e^{-\Lambda m r} P_a;\ P_a := a_0 + a_1 r + \cdots + a_n r^n, \tag{C.165}$$

$$B := r^s e^{-\Lambda m r} P_b;\ P_b := b_0 + b_1 r + \cdots + b_n r^n, \tag{C.166}$$

$$C := r^s e^{-\Lambda m r} P_c;\ P_c := c_0 + c_1 r + \cdots + c_n r^n, \tag{C.167}$$

$$D := r^s e^{-\Lambda m r} P_d;\ P_d := d_0 + d_1 r + \cdots + d_n r^n. \tag{C.168}$$

And since $U=-\sqrt{\sin\theta}\cos\frac{\theta}{2}$ and $V=-\sqrt{\sin\theta}\sin\frac{\theta}{2}$, we obtain:

$$\phi = r^{s-1}e^{-\Lambda mr}e^{-\frac{\varphi}{2}i_3}Me^{\frac{\varphi}{2}i_3}e^{-E\mathrm{x}^0 i_3},$$
$$M = \begin{pmatrix} \frac{P_c-P_a}{2} - \frac{P_a+P_c}{2}\cos\theta & \frac{P_b^*+P_d^*}{2}\sin\theta \\ -\frac{P_a+P_c}{2}\sin\theta & \frac{P_b^*-P_d^*}{2} - \frac{P_b^*+P_d^*}{2}\cos\theta \end{pmatrix} \tag{C.169}$$

We then have:

$$M = \begin{pmatrix} \frac{P_c-P_a}{2} & 0 \\ 0 & \frac{P_b^*-P_d^*}{2} \end{pmatrix} - \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \frac{P_a+P_c}{2} & 0 \\ 0 & \frac{P_b^*+P_d^*}{2} \end{pmatrix}$$
$$= M_- - e^{-\theta i_2}M_+ \tag{C.170}$$
$$M_- := (P_c-P_a)\frac{1+\sigma_3}{4} + (P_b^*-P_d^*)\frac{1-\sigma_3}{4}, \tag{C.171}$$
$$M_+ := (P_c+P_a)\frac{1+\sigma_3}{4} + (P_b^*+P_d^*)\frac{1-\sigma_3}{4}. \tag{C.172}$$

That gives:

$$\phi = r^{s-1}e^{-\Lambda mr}\Big(M_- - \frac{\vec{\mathrm{x}}}{r}\sigma_3 M_+\Big)e^{-E\mathrm{x}^0 i_3}, \tag{C.173}$$
$$\widehat{\phi} = r^{s-1}e^{-\Lambda mr}\Big(\widehat{M}_- - \frac{\vec{\mathrm{x}}}{r}\sigma_3 \widehat{M}_+\Big)e^{-E\mathrm{x}^0 i_3}. \tag{C.174}$$

With $\kappa=1$ and $\lambda=-1/2$ we have:

$$\phi = r^{s-1}e^{-\Lambda mr}e^{-\frac{\varphi}{2}i_3}(-i_2)Me^{\frac{-\varphi}{2}i_3}e^{-E\mathrm{x}^0 i_3},$$
$$M = \begin{pmatrix} \frac{P_c-P_a}{2} + \frac{P_a+P_c}{2}\cos\theta & -\frac{P_b^*+P_d^*}{2}\sin\theta \\ \frac{P_a+P_c}{2}\sin\theta & \frac{P_b^*-P_d^*}{2} + \frac{P_b^*+P_d^*}{2}\cos\theta \end{pmatrix}$$
$$\phi = r^{s-1}e^{-\Lambda mr}\Big[-i_2 M_- - \frac{\vec{\mathrm{x}}}{r}\sigma_1 M_+\Big]e^{-E\mathrm{x}^0 i_3}. \tag{C.175}$$

If $\kappa=-1$ and $\lambda=1/2$ we obtain:

$$C(\theta) = \sum_{n=0}^{\infty}\frac{(\frac{1}{2}+1-\frac{1}{2})_n(\frac{1}{2}-1+\frac{1}{2})_n}{(\frac{1}{2}+\frac{1}{2})_n n!}\sin^{2n}(\frac{\theta}{2})$$
$$= \sum_{n=0}^{\infty}\frac{(1)_n(0)_n}{(1)_n n!}\sin^{2n}\Big(\frac{\theta}{2}\Big) = 1, \tag{C.176}$$
$$C'(\theta)=0;\ U=\sqrt{\sin\theta}\Big[0-(-1)\cos\frac{\theta}{2}\Big] = \sqrt{\sin\theta}\cos\frac{\theta}{2}, \tag{C.177}$$
$$V=\sqrt{\sin\theta}\Big[0+(-1)\sin\frac{\theta}{2}\Big] = -\sqrt{\sin\theta}\sin\frac{\theta}{2}. \tag{C.178}$$
$$U^2=\sin\theta\cos^2\frac{\theta}{2};\ V^2=\sin\theta\sin^2\frac{\theta}{2},$$
$$U^2+V^2=\sin\theta;\ U^2-V^2=\sin\theta\cos\theta;\ 2UV=-\sin^2\theta, \tag{C.179}$$
$$2d_0^0 = (AA^*+DD^*)\sin\theta\cos^2\frac{\theta}{2} + (BB^*+CC^*)\sin\theta\sin^2\frac{\theta}{2}, \tag{C.180}$$

$$\begin{aligned}
\mathrm{J}^0 &= \mathrm{D}^0_0 = \frac{d^0_0}{r^2 \sin\theta} \\
&= \frac{1}{2r^2}\Big[(AA^* + DD^*)\cos^2\frac{\theta}{2} + (BB^* + CC^*)\sin^2\frac{\theta}{2}\Big], \qquad \text{(C.181)} \\
\mathbf{J}^0 &= \frac{m}{2kr^2}\Big[(\mathbf{l}AA^* + \mathbf{r}DD^*)\cos^2\frac{\theta}{2} + (\mathbf{r}BB^* + \mathbf{l}CC^*)\sin^2\frac{\theta}{2}\Big]. \qquad \text{(C.182)}
\end{aligned}$$

We have also:

$$\begin{aligned}
\phi &= r^{s-1}e^{-\Lambda mr}e^{-\frac{\varphi}{2}i_3}e^{-\frac{\theta}{2}i_2}\begin{pmatrix} P_a\cos\frac{\theta}{2} & P_b^*\sin\frac{\theta}{2} \\ -P_c\sin\frac{\theta}{2} & P_d^*\cos\frac{\theta}{2}\end{pmatrix} e^{(\frac{\varphi}{2}-E\mathrm{x}^0)i_3} \\
&= r^{s-1}e^{-\Lambda mr}e^{-\frac{\varphi}{2}i_3}(M_+ - e^{-\theta i_2}M_-)e^{(\frac{\varphi}{2}-E\mathrm{x}^0)i_3} \\
&= r^{s-1}e^{-\Lambda mr}(M_+ - \frac{\vec{\mathrm{x}}}{r}\sigma_3 M_-)e^{-E\mathrm{x}^0 i_3}. \qquad \text{(C.183)}
\end{aligned}$$

Similarly if $\kappa = -1$ and $\lambda = -1/2$ we again have $C = 1$ and $C' = 0$, and thus (C.53) gives:

$$U = \sqrt{\sin\theta}\Big[0 + (-1)\sin\frac{\theta}{2}\Big] = -\sqrt{\sin\theta}\sin\frac{\theta}{2}, \qquad \text{(C.184)}$$

$$V = \sqrt{\sin\theta}\Big[0 + (-1)\cos\frac{\theta}{2}\Big] = -\sqrt{\sin\theta}\cos\frac{\theta}{2}. \qquad \text{(C.185)}$$

That implies:

$$\begin{aligned}
U^2 &= \sin\theta\sin^2\frac{\theta}{2};\ V^2 = \sin\theta\cos^2\frac{\theta}{2} \\
U^2 + V^2 &= \sin\theta;\ U^2 - V^2 = -\sin\theta\cos\theta;\ 2UV = \sin^2\theta. \qquad \text{(C.186)} \\
2d^0_0 &= (AA^* + DD^*)\sin\theta\sin^2\frac{\theta}{2} + (BB^* + CC^*)\sin\theta\cos^2\frac{\theta}{2}, \\
\mathrm{J}^0 &= \frac{1}{2r^2}\Big[(AA^* + DD^*)\sin^2\frac{\theta}{2} + (BB^* + CC^*)\cos^2\frac{\theta}{2}\Big], \qquad \text{(C.187)} \\
\mathbf{J}^0 &= \frac{m}{2kr^2}\Big[(\mathbf{l}AA^* + \mathbf{r}DD^*)\sin^2\frac{\theta}{2} + (\mathbf{r}BB^* + \mathbf{l}CC^*)\cos^2\frac{\theta}{2}\Big].
\end{aligned}$$

These values are those of $\mathbf{n}p_{1/2}$ states ($n = \mathbf{n} - 1$, $n > 0$, $\kappa = -1$, $j = 1/2$). The ϕ wave satsfies, with $\lambda = -1/2$:

$$\begin{aligned}
\phi &= r^{s-1}e^{-\Lambda mr}e^{-\frac{\varphi}{2}i_3}e^{-\frac{\theta}{2}i_2}\begin{pmatrix} -P_a\sin\frac{\theta}{2} & P_b^*\cos\frac{\theta}{2} \\ -P_c\cos\frac{\theta}{2} & -P_d^*\sin\frac{\theta}{2}\end{pmatrix} e^{(-\frac{\varphi}{2}-E\mathrm{x}^0)i_3} \\
&= r^{s-1}e^{-\Lambda mr}e^{-\frac{\varphi}{2}i_3}(M_+ + e^{-\theta i_2}M_-)e^{(-\frac{\varphi}{2}-E\mathrm{x}^0)i_3} \\
&= r^{s-1}e^{-\Lambda mr}(i_2 M_+ + \frac{\vec{\mathrm{x}}}{r}\sigma_1 M_-)e^{-E\mathrm{x}^0 i_3}. \qquad \text{(C.188)}
\end{aligned}$$

Appendix D

Miscellaneous

D.1 Gauge invariance SU(2) of the quarks

D.1.1 Group generated by $\underline{P}_1$

We have in this case

$$\Psi_L = \underline{P}_+(\Psi);\ \underline{P}_1(\Psi_L) = \Gamma_{0123}\Psi_L\Gamma_{35};\ C = \cos(\theta);\ S = \sin(\theta), \tag{D.1}$$

$$\Psi'_L = [\exp(\theta\underline{P}_1)](\Psi_L) = C\Psi_L + S\Gamma_{0123}\Psi_L\Gamma_{35}, \tag{D.2}$$

$$W'^1_\mu = W^1_\mu - \frac{2}{g_2}\partial_\mu\theta. \tag{D.3}$$

This gives:

$$\begin{pmatrix} L'^n & \widetilde{L}'^{3+n} \\ \overline{L}'^{3+n} & -\widehat{L}'^n \end{pmatrix} = C\begin{pmatrix} L^n & \widetilde{L}^{3+n} \\ \overline{L}^{3+n} & -\widehat{L}^n \end{pmatrix} + S\begin{pmatrix} i\widetilde{L}^{3+n} & iL^n \\ -i\overline{L}^{3+n} & i\widehat{L}^n \end{pmatrix}.\ n = 2,3,4, \tag{D.4}$$

$$L'^n = CL^n + iS\widetilde{L}^{3+n}, \tag{D.5}$$

$$\widetilde{L}'^{3+n} = C\widetilde{L}^{3+n} + iSL^n. \tag{D.6}$$

We now let:

$$2L^nL^{3+n} = D_L^{n\,3+n} - id_L^{n\,3+n}. \tag{D.7}$$

We deduce for the left currents:

$$D_L^{n\,3+n} = L^nL^{3+n} + \widetilde{L}^{3+n}\widetilde{L}^n;\ d_L^{n\,3+n} = iL^nL^{3+n} - i\widetilde{L}^{3+n}\widetilde{L}^n, \tag{D.8}$$

$$2D'^n_L = D^n_L + D^{3+n}_L + \cos(2\theta)(D^n_L - D^{3+n}_L) - \sin(2\theta)d_L^{n\,3+n}, \tag{D.9}$$

$$2D'^{3+n} = D^n_L + D^{3+n}_L - \cos(2\theta)(D^n_L - D^{3+n}_L) + \sin(2\theta)d_L^{n\,3+n}. \tag{D.10}$$

Adding and subtracting these equations we get:

$$D_L'^n + D_L'^{3+n} = D_L^n + D_L^{3+n}, \tag{D.11}$$
$$D_L'^{3+n} - D_L'^n = \cos(2\theta)(D_L^{3+n} - D_L^n) + \sin(2\theta) d_L^{n\,3+n}. \tag{D.12}$$

From these equations, we obtain the conservation of the total current J_q, and also the difference between the left currents and the right currents. By bringing together these equations and (2.124) we may see that they are compatible with:

$$W^2 = d_L^{n\,3+n};\ W^3 = D_L^{3+n} - D_L^n. \tag{D.13}$$

D.1.2 Groups generated by $\underline{P}_2$ and $\underline{P}_3$

The calculation is completely similar to the previous section. In both cases we obtain the value of W^1 as only additional result. And as before, this value depends on the integer n. Also we are going to use a double system of indices: we will denote as W_n^j the potentials previously denoted as W^j acting on L^n. We will then have:

$$W_n^1 = D_L^{n\,3+n};\ W_n^2 = d_L^{n\,3+n};\ W_n^3 = D_L^{3+n} - D_L^n. \tag{D.14}$$

As in the previous section the sum of the left currents and the sum of the rights currents are invariant, which implies the conservation of the J_q current, and thus of $m\mathrm{v}_q$. This makes the system formed by the twelve wave equations (6 right and 6 left) invariant under the $SU(2)$ group.

D.2 Gauge invariance under SU(3)

We use the following transformation:

$$\Psi' = [\exp(\theta\Lambda_1)](\Psi),$$
$$\Psi'^2 = C\Psi^2 + S\mathbf{i}\Psi^3;\ C = \cos(\theta);\ S = \sin(\theta), \tag{D.15}$$
$$\Psi'^3 = C\Psi^3 + S\mathbf{i}\Psi^2;\ \Psi'^1 = \Psi^1;\ \Psi'^4 = \Psi^4. \tag{D.16}$$

Here we can forget about Ψ^1 and Ψ^4 which do not vary. The gauge invariance means that the system:

$$\boldsymbol{\partial}\Psi^2 = -\frac{g_3}{2}\mathbf{G}^1\mathbf{i}\Psi^3 + m\mathbf{v}_q\Psi^2\gamma_{12},$$
$$\boldsymbol{\partial}\Psi^3 = -\frac{g_3}{2}\mathbf{G}^1\mathbf{i}\Psi^2 + m\mathbf{v}_q\Psi^3\gamma_{12}. \tag{D.17}$$

must be equivalent to the system:

$$\boldsymbol{\partial}\Psi'^2 = -\frac{g_3}{2}\mathbf{G}'^1\mathbf{i}\Psi'^3 + m\mathbf{v}_q\Psi'^2\gamma_{12},$$
$$\boldsymbol{\partial}\Psi'^3 = -\frac{g_3}{2}\mathbf{G}'^1\mathbf{i}\Psi'^2 + m\mathbf{v}_q\Psi'^3\gamma_{12}. \tag{D.18}$$

Using (D.15) and (D.16) the system (D.17) is equivalent to (D.18) if and only if:

$$\mathbf{G'}^1 = \mathbf{G}^1 - \frac{2}{g_3}\boldsymbol{\partial}\theta;\ h'^1_1 = h^1_1 - \boldsymbol{\partial}\theta. \tag{D.19}$$

The equality (D.15) is equivalent to the following system:

$$L'^2 = CL^2 + iSL^3;\ \widetilde{L}'^5 = C\widetilde{L}^5 + iS\widetilde{L}^6, \tag{D.20}$$
$$R'^2 = CR^2 + iSR^3;\ \widetilde{R}'^5 = C\widetilde{R}^5 + iS\widetilde{R}^6. \tag{D.21}$$

Meanwhile the equality (D.16) is equivalent to the system:

$$L'^3 = CL^3 + iSL^2;\ \widetilde{L}'^6 = C\widetilde{L}^6 + iS\widetilde{L}^5, \tag{D.22}$$
$$R'^3 = CR^3 + iSR^2;\ \widetilde{R}'^6 = C\widetilde{R}^6 + iS\widetilde{R}^5. \tag{D.23}$$

These systems may be brought together: we obtain four systems with the same structure:

$$L'^2 = CL^2 + iSL^3;\ L'^3 = CL^3 + iSL^2, \tag{D.24}$$
$$R'^2 = CR^2 + iSR^3;\ R'^3 = CR^3 + iSR^2, \tag{D.25}$$
$$\widetilde{L}'^5 = C\widetilde{L}^5 + iS\widetilde{L}^6;\ \widetilde{L}'^6 = C\widetilde{L}^6 + iS\widetilde{L}^5, \tag{D.26}$$
$$\widetilde{R}'^6 = C\widetilde{R}^6 + iS\widetilde{R}^5;\ \widetilde{R}'^6 = C\widetilde{R}^6 + iS\widetilde{R}^5. \tag{D.27}$$

These systems have the same form as those of the left waves for the weak interaction. We can hence carry out similar calculations as in D.1.1. For the left waves of the d quark with color r or g, we consider the currents:

$$D^2_L = L^2\widetilde{L}^2;\ D^3_L = L^3\widetilde{L}^3;\ D^{2\,3}_L - id^{2\,3}_L = 2L^2\widetilde{L}^3, \tag{D.28}$$
$$D^{2\,3}_L + id^{2\,3}_L = 2L^3\widetilde{L}^2;\ D^{2\,3}_L = L^2\widetilde{L}^3 + L^3\widetilde{L}^2;\ d^{2\,3}_L = iL^2\widetilde{L}^3 - iL^3\widetilde{L}^2. \tag{D.29}$$

We then get:

$$D'^{2\,3}_L = D^{2\,3}_L;\ D'^2_L + D'^3_L = D^2_L + D^3_L, \tag{D.30}$$
$$d'^{2\,3}_L = \cos(2\theta)d^{2\,3}_L - \sin(2\theta)(D^3_L - D^2_L), \tag{D.31}$$
$$D'^3_L - D'^2_L = \cos(2\theta)(D^3_L - D^2_L) + \sin(2\theta)d^{2\,3}_L. \tag{D.32}$$

A comparison with the rotation made on the potentials by the gauge transformation indicates that we can have:

$$\mathrm{h}^1_1 = \frac{g_3}{2}D^{2\,3}_L;\ \mathrm{h}^2_1 = \frac{g_3}{2}d^{2\,3}_L;\ \mathrm{h}^3_1 = \frac{g_3}{2}(D^3_L - D^2_L). \tag{D.33}$$

For the right waves of the d quark with color r or g we consider the currents:

$$D^2_R = R^2\widetilde{R}^2;\ D^3_R = R^3\widetilde{R}^3;\ D^{2\,3}_R - id^{2\,3}_R = 2R^2\widetilde{R}^3, \tag{D.34}$$
$$D^{2\,3}_R + id^{2\,3}_R = 2R^3\widetilde{R}^2;\ D^{2\,3}_R = R^2\widetilde{R}^3 + R^3\widetilde{R}^2;\ d^{2\,3}_R = iR^2\widetilde{R}^3 - iR^3\widetilde{R}^2. \tag{D.35}$$

We thus get :

$$D'^{2\,3}_R = D^{2\,3}_R;\ D'^2_R + D'^3_R = D^2_R + D^3_R, \tag{D.36}$$

$$d'^{2\,3}_R = \cos(2\theta) d^{2\,3}_R - \sin(2\theta)(D^3_R - D^2_R), \tag{D.37}$$

$$D'^3_R - D'^2_R = \cos(2\theta)(D^3_R - D^2_R) + \sin(2\theta) d^{2\,3}_R. \tag{D.38}$$

A comparison with the rotation made on the potentials by the gauge transformation indicates that we can have:

$$\mathrm{h}^1_1 = \frac{g_3}{2} D^{2\,3}_R;\ \mathrm{h}^2_1 = \frac{g_3}{2} d^{2\,3}_R;\ \mathrm{h}^3_1 = \frac{g_3}{2}(D^3_R - D^2_R). \tag{D.39}$$

Hence, here we again see the dependence of the potentials on the wave which they are acting on. We thus note:

$$\mathrm{h}^{d\,1}_{L1} = \frac{g_3}{2} D^{2\,3}_L;\ \mathrm{h}^{d\,2}_{L1} = \frac{g_3}{2} d^{2\,3}_L;\ \mathrm{h}^{d\,3}_{L1} = \frac{g_3}{2}(D^3_L - D^2_L), \tag{D.40}$$

$$\mathrm{h}^{d\,1}_{R1} = \frac{g_3}{2} D^{2\,3}_R;\ \mathrm{h}^{d\,2}_{R1} = \frac{g_3}{2} d^{2\,3}_R;\ \mathrm{h}^{d\,3}_{R1} = \frac{g_3}{2}(D^3_R - D^2_R). \tag{D.41}$$

For the left wave of the u quark with color r or g we consider the currents:

$$D^5_L = \widetilde{L}^5 L^5;\ D^6_L = \widetilde{L}^6 L^6;\ D^{5\,6}_L - i d^{5\,6}_L = 2\widetilde{L}^5 L^6, \tag{D.42}$$

$$D^{5\,6}_L + i d^{5\,6}_L = 2\widetilde{L}^6 L^5;\ D^{5\,6}_L = \widetilde{L}^5 L^6 + \widetilde{L}^6 L^5;\ d^{5\,6}_L = i\widetilde{L}^5 L^6 - i\widetilde{L}^6 L^5. \tag{D.43}$$

We thus obtain:

$$D'^{5\,6}_L = D^{5\,6}_L;\ D'^5_L + D'^6_L = D^5_L + D^6_L, \tag{D.44}$$

$$d'^{5\,6}_L = \cos(2\theta) d^{5\,6}_L - \sin(2\theta)(D^6_L - D^5_L), \tag{D.45}$$

$$D'^6_L - D'^5_L = \cos(2\theta)(D^6_L - D^5_L) + \sin(2\theta) d^{5\,6}_L. \tag{D.46}$$

A comparison with the rotation made on the potentials by the gauge transformation indicates that we can have:

$$\mathrm{h}^{u\,1}_{L1} = \frac{g_3}{2} D^{5\,6}_L;\ \mathrm{h}^{u\,2}_{L1} = \frac{g_3}{2} d^{5\,6}_L;\ \mathrm{h}^{u\,3}_{L1} = \frac{g_3}{2}(D^6_L - D^5_L). \tag{D.47}$$

For the right wave of the u quark with color r or g we consider the currents:

$$D^5_R = \widetilde{R}^5 R^5;\ D^6_R = \widetilde{R}^6 R^6;\ D^{5\,6}_R - i d^{5\,6}_R = 2\widetilde{R}^5 R^6, \tag{D.48}$$

$$D^{5\,6}_R + i d^{5\,6}_R = 2\widetilde{R}^6 R^5;\ D^{5\,6}_R = \widetilde{R}^5 R^6 + \widetilde{R}^6 R^5;\ d^{5\,6}_R = i\widetilde{R}^5 R^6 - i\widetilde{R}^6 R^5. \tag{D.49}$$

We then get:

$$D'^{5\,6}_R = D^{5\,6}_R;\ D'^5_R + D'^6_R = D^5_R + D^6_R, \tag{D.50}$$

$$d'^{5\,6}_R = \cos(2\theta) d^{5\,6}_R - \sin(2\theta)(D^6_R - D^5_R), \tag{D.51}$$

$$D'^6_R - D'^5_R = \cos(2\theta)(D^6_R - D^5_R) + \sin(2\theta) d^{5\,6}_R. \tag{D.52}$$

Finally, a comparison with the rotation made on the potentials by the gauge transformation indicates that we can have:

$$\mathrm{h}^{u\,1}_{R1} = \frac{g_3}{2} D^{5\,6}_R;\ \mathrm{h}^{u\,2}_{R1} = \frac{g_3}{2} d^{5\,6}_R;\ \mathrm{h}^{u\,3}_{R1} = \frac{g_3}{2}(D^6_R - D^5_R). \tag{D.53}$$

D.3 Simplification of the wave equations

With (D.7) and (D.14) we have:

$$(W_n^1 + iW_n^2)\overline{L}^{3+n} = (D_L^{n\,3+n} + id_L^{n\,3+n})\overline{L}^{3+n} = 2\widetilde{L}^{3+n}\widetilde{L}^n\overline{L}^{3+n}, \quad \text{(D.54)}$$

$$\widetilde{L}^n\overline{L}^{3+n} = 2\begin{pmatrix} 0 & 0 \\ -\eta_2^n & \eta_1^n \end{pmatrix}\begin{pmatrix} \eta_1^{3+n} & 0 \\ \eta_2^{3+n} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2(-\eta_2^n\eta_1^{3+n} + \eta_1^n\eta_2^{3+n}) & 0 \end{pmatrix}.$$

And given that:

$$2(-\eta_2^n\eta_1^{3+n} + \eta_1^n\eta_2^{3+n}) = 2\widehat{\eta}^{n\dagger}\eta^{3+n} = \overline{s}_2^{3+n\,n}, \quad \text{(D.55)}$$

$$\widetilde{L}^{3+n}\widetilde{L}^n\overline{L}^{3+n} = \sqrt{2}\begin{pmatrix} 0 & \overline{\eta}_2^{3+n} \\ 0 & \overline{\eta}_1^{3+n} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \overline{s}_2^{3+n\,n} & 0 \end{pmatrix} = \overline{s}_2^{3+n\,n}\widetilde{L}^{3+n}\sigma_1.$$

We also have:

$$W_n^3\widehat{L}^n = (D^{3+n} - D^n)\widehat{L}^n = \widetilde{L}^{3+n}L^{3+n}\widehat{L}^n - L^n\widetilde{L}^n\widehat{L}^n = \widetilde{L}^{3+n}L^{3+n}\widehat{L}^n, \quad \text{(D.56)}$$

$$L^{3+n}\widehat{L}^n = \overline{\widetilde{L}^n\overline{L}^{3+n}} = \overline{\overline{s}_2^{3+n\,n}\frac{\sigma_1 - i\sigma_2}{2}} = -\overline{s}_2^{3+n\,n}\frac{\sigma_1 - i\sigma_2}{2} = -\widetilde{L}^n\overline{L}^{3+n},$$

$$W_n^3\widehat{L}^n = \widetilde{L}^{3+n}L^{3+n}\widehat{L}^n = -\widetilde{L}^{3+n}\widetilde{L}^n\overline{L}^{3+n}. \quad \text{(D.57)}$$

Thus we obtain:

$$\begin{aligned}(W_n^1 + iW_n^2)\overline{L}^{3+n} - W_n^3\widehat{L}^n &= 2\widetilde{L}^{3+n}\widetilde{L}^n\overline{L}^{3+n} - (-\widetilde{L}^{3+n}\widetilde{L}^n\overline{L}^{3+n}) \\ &= 3\widetilde{L}^{3+n}\widetilde{L}^n\overline{L}^{3+n} = -3W_n^3\widehat{L}^n. \quad \text{(D.58)}\end{aligned}$$

Furthermore:

$$(W_n^1 - iW_n^2)\widehat{L}^n = (D_L^{n\,3+n} - id_L^{n\,3+n})\widehat{L}^n = 2L^nL^{3+n}\widehat{L}^n, \quad \text{(D.59)}$$

$$L^{3+n}\widehat{L}^n = \overline{\widetilde{L}^n\overline{L}^{3+n}} = -\widetilde{L}^n\overline{L}^{3+n},$$

$$(W_n^1 - iW_n^2)\widehat{L}^n = 2L^n(-\widetilde{L}^n\overline{L}^{3+n}) = -2L^n\widetilde{L}^n\overline{L}^{3+n} = -2D_L^n\overline{L}^{3+n}, \quad \text{(D.60)}$$

$$W_n^3\overline{L}^{3+n} = (D_n^{3+n} - D_L^n)\overline{L}^{3+n} = -D_L^n\overline{L}^{3+n},$$

$$(W_n^1 - iW_n^2)\widehat{L}^n + W_n^3\overline{L}^{3+n} = -3D_L^n\overline{L}^{3+n} = 3W_n^3\overline{L}^{3+n}. \quad \text{(D.61)}$$

For the gauge group of chromodynamics we have the same sort of simplification, we will see this in detail for one of the four cases, that of the left wave of the d quark. With the gauge transformation generated by Γ_1 and with an angle θ we have:

$$(\mathrm{h}_{L1}^{d\,1} + i\mathrm{h}_{L1}^{d\,2})\widehat{L}^3 - \mathrm{h}_{L1}^{d\,3}\widehat{L}^2 = \frac{g_3}{2}[(D_L^{23} + id_L^{23})\widehat{L}^3 - (D_L^3 - D_L^2)\widehat{L}^2], \quad \text{(D.62)}$$

$$L'^2 = CL^2 + iS\widetilde{L}^3;\ C = \cos(\theta);\ S = \sin(\theta), \quad \text{(D.63)}$$

$$\widetilde{L}'^3 = C\widetilde{L}^3 + iSL^2. \quad \text{(D.64)}$$

Only the indices change in comparison with (D.5) and (D.6), where for $n = 2$ we have the same relations with indices 2 and 5 instead of the current indices 2 and 3. And so we can employ the same procedure that carried out for the weak interactions. We finally have:

$$\mathrm{h}^{d\,1}_{L1} = \frac{g_3}{2} D^{23}_L;\ \mathrm{h}^{d\,2}_{L1} = \frac{g_3}{2} d^{23}_L;\ \mathrm{h}^{d\,3}_{L1} = \frac{g_3}{2}(D^3_L - D^2_L), \tag{D.65}$$

$$(\mathrm{h}^{d\,1}_{L1} + i\mathrm{h}^{d\,2}_{L1})\widehat{L}^3 - \mathrm{h}^{d\,3}_{L1}\widehat{L}^2 = -3\mathrm{h}^{d\,3}_{L1}\widehat{L}^2 = -\frac{3g_3}{2} D^3_L \widehat{L}^2, \tag{D.66}$$

$$(\mathrm{h}^{d\,1}_{L1} - i\mathrm{h}^{d\,2}_{L1})\widehat{L}^2 + \mathrm{h}^{d\,3}_{L1}\widehat{L}^3 = 3\mathrm{h}^{d\,3}_{L1}\widehat{L}^3 = -\frac{3g_3}{2} D^2_L \widehat{L}^3. \tag{D.67}$$

We obtain similar equations for the other indices, for the right waves and for the u quark, this allows us to simplify the wave equations.

D.3.1 Gauge terms of the Lagrangian density

We consider the S part of the Lagrangian density $\mathcal{L}^+$ that gives the gauge terms acting on the waves of the quarks:

$$\begin{aligned} S &= \sum_{n=2}^{4} \begin{bmatrix} -i\frac{\mathrm{m}}{q_1}\eta^{n\dagger}\sigma^\mu(ig^1_{n\mu})\eta^n - i\frac{\mathrm{m}}{q_2}\xi^{n\dagger}\widehat{\sigma}^\mu(ig^2_{n\mu})\xi^n \\ -i\frac{\mathrm{m}}{q_3}\eta^{3+n\dagger}\sigma^\mu(ig^3_{n\mu})\eta^{3+n} - i\frac{\mathrm{m}}{q_4}\xi^{3+n\dagger}\widehat{\sigma}^\mu(ig^4_{n\mu})\xi^{3+n} \end{bmatrix} \\ &= \sum_{n=2}^{4} \begin{bmatrix} \frac{\mathrm{m}}{q_1}g^1_{n\mu}\eta^{n\dagger}\sigma^\mu\eta^n + \frac{\mathrm{m}}{q_2}g^2_{n\mu}\xi^{n\dagger}\widehat{\sigma}^\mu\xi^n \\ +\frac{\mathrm{m}}{q_3}g^3_{n\mu}\eta^{3+n\dagger}\sigma^\mu\eta^{3+n} + \frac{\mathrm{m}}{q_4}g^4_{n\mu}\xi^{3+n\dagger}\widehat{\sigma}^\mu\xi^{3+n} \end{bmatrix}. \end{aligned} \tag{D.68}$$

We thus have:

$$\begin{aligned} &S = S^1 + S^2 + S^3 + S^4;\ S^1 = \frac{\mathrm{m}}{q_1}\sum_{n=2}^{4} g^1_{n\mu}D^{n\mu}_L, \\ &S^2 = \frac{\mathrm{m}}{q_2}\sum_{n=2}^{4} g^2_{n\mu}D^{n\mu}_R;\ S^3 = \frac{\mathrm{m}}{q_3}\sum_{n=2}^{4} g^3_{n\mu}D^{3+n\mu}_L;\ S^4 = \frac{\mathrm{m}}{q_4}\sum_{n=2}^{4} g^4_{n\mu}D^{3+n\mu}_R. \end{aligned} \tag{D.69}$$

With (3.142) the S^1 term becomes:

$$\begin{aligned} \frac{q_1}{\mathrm{m}}S^1 &= \Big(-\frac{\mathrm{b}_\mu}{3} + 3\mathrm{w}^3_{2\mu} - 3\mathrm{h}^{d3}_{L3\mu} + 3\mathrm{h}^{d3}_{L1\mu}\Big)D^{2\mu}_L \\ &+ \Big(-\frac{\mathrm{b}_\mu}{3} + 3\mathrm{w}^3_{3\mu} - 3\mathrm{h}^{d3}_{L1\mu} + 3\mathrm{h}^{d3}_{L2\mu}\Big)D^{3\mu}_L \\ &+ \Big(-\frac{\mathrm{b}_\mu}{3} + 3\mathrm{w}^3_{4\mu} - 3\mathrm{h}^{d3}_{L2\mu} + 3\mathrm{h}^{d3}_{L3\mu}\Big)D^{4\mu}_L. \end{aligned} \tag{D.70}$$

Grouping together similar terms we get:

$$\begin{aligned} &\frac{q_1}{\mathrm{m}}S^1 = -\frac{1}{3}\mathrm{b}\cdot(D^2_L + D^3_L + D^4_L) + 3(\mathrm{w}^3_2\cdot D^2_L + \mathrm{w}^3_3\cdot D^3_L + \mathrm{w}^3_4\cdot D^4_L) \\ &- 3(\mathrm{h}^{d3}_{L3}\cdot D^2_L + \mathrm{h}^{d3}_{L1}\cdot D^3_L + \mathrm{h}^{d3}_{L2}\cdot D^4_L) + 3(\mathrm{h}^{d3}_{L1}\cdot D^2_L + \mathrm{h}^{d3}_{L2}\cdot D^3_L + \mathrm{h}^{d3}_{L3}\cdot D^4_L). \end{aligned} \tag{D.71}$$

And with (3.126) we have:

$$\begin{aligned}
&3(\mathrm{w}_2^3 \cdot D_L^2 + \mathrm{w}_3^3 \cdot D_L^3 + \mathrm{w}_4^3 \cdot D_L^4) \\
&= \frac{g_2}{2}[(D_L^5 - D_L^2) \cdot D_L^2 + (D_L^6 - D_L^3) \cdot D_L^3 + (D_L^7 - D_L^4) \cdot D_L^4 \\
&= \frac{g_2}{2}(D_L^5 \cdot D_L^2 + D_L^6 \cdot D_L^3 + D_L^7 \cdot D_L^4),
\end{aligned} \tag{D.72}$$

and since the chiral currents are on the light cone, their space-time length is null. Next with (D.40) we have, $\mathrm{h}^{d\,1}_{L1} = \frac{g_3}{2} D_L^{2\,3}$, $\mathrm{h}^{d\,2}_{L1} = \frac{g_3}{2} d_L^{2\,3}$ and $\mathrm{h}^{d\,3}_{L1} = \frac{g_3}{2}(D_L^3 - D_L^2)$. We thus get:

$$\begin{aligned}
&3(\mathrm{h}^{d3}_{L1} \cdot D_L^2 + \mathrm{h}^{d3}_{L2} \cdot D_L^3 + \mathrm{h}^{d3}_{L3} \cdot D_L^4) \\
&= 3\frac{g_3}{2}[(D_L^3 - D_L^2) \cdot D_L^2 + (D_L^4 - D_L^3) \cdot D_L^3 + (D_L^2 - D_L^4) \cdot D_L^4] \\
&= 3\frac{g_3}{2}(D_L^3 \cdot D_L^2 + D_L^4 \cdot D_L^3 + D_L^2 \cdot D_L^4).
\end{aligned} \tag{D.73}$$

Similarly we have:

$$\begin{aligned}
&-3(\mathrm{h}^{d3}_{L3} \cdot D_L^2 + \mathrm{h}^{d3}_{L1} \cdot D_L^3 + \mathrm{h}^{d3}_{L2} \cdot D_L^4) \\
&= -3\frac{g_3}{2}[(D_L^2 - D_L^4) \cdot D_L^2 + (D_L^3 - D_L^2) \cdot D_L^3 + (D_L^4 - D_L^3) \cdot D_L^4] \\
&= 3\frac{g_3}{2}(D_L^4 \cdot D_L^2 + D_L^2 \cdot D_L^3 + D_L^3 \cdot D_L^4) \\
&= 3\frac{g_3}{2}(D_L^3 \cdot D_L^2 + D_L^4 \cdot D_L^3 + D_L^2 \cdot D_L^4).
\end{aligned} \tag{D.74}$$

And if we use the sums of currents:

$$\begin{aligned}
&S_L^{25} = D_L^2 + D_L^5;\ S_L^{36} = D_L^3 + D_L^6;\ S_L^{47} = D_L^4 + D_L^7, \\
&S_L^d = D_L^2 + D_L^3 + D_L^4,
\end{aligned} \tag{D.75}$$

we have:

$$(S_L^{25})^2 = (D_L^2)^2 + (D_L^5)^2 + 2D_L^2 \cdot D_L^5 = 2D_L^2 \cdot D_L^5, \tag{D.76}$$

$$\begin{aligned}
(S_L^d)^2 &= (D_L^2)^2 + (D_L^3)^2 + (D_L^4)^2 + 2D_L^2 \cdot D_L^3 + 2D_L^3 \cdot D_L^4 + 2D_L^4 \cdot D_L^2 \\
&= 2(D_L^2 \cdot D_L^3 + D_L^3 \cdot D_L^4 + D_L^4 \cdot D_L^2).
\end{aligned} \tag{D.77}$$

And we thus get:

$$S^1 = \frac{\mathrm{m}}{q_1}\Big[-\frac{g_1}{6} B \cdot S_L^d + \frac{g_2}{4}\big[(S_L^{25})^2 + (S_L^{36})^2 + (S_L^{47})^2\big] + \frac{3g_3}{2}(S_L^d)^2\Big]. \tag{D.78}$$

Next for the right waves of the d quark we have:

$$\begin{aligned}
&\frac{q_2}{\mathrm{m}} S^2 = \frac{2}{3}\mathrm{b} \cdot (D_R^2 + D_R^3 + D_R^4) \\
&+ 3(\mathrm{h}^{d3}_{R3} \cdot D_R^2 + \mathrm{h}^{d3}_{R1} \cdot D_R^3 + \mathrm{h}^{d3}_{R2} \cdot D_R^4) - 3(\mathrm{h}^{d3}_{R1} \cdot D_R^2 + \mathrm{h}^{d3}_{R2} \cdot D_R^3 + \mathrm{h}^{d3}_{R3} \cdot D_R^4).
\end{aligned} \tag{D.79}$$

We now use the sum of the right currents:

$$S_R^d = D_R^2 + D_R^3 + D_R^4, \tag{D.80}$$

And we get:

$$S^2 = \frac{\mathrm{m}}{q_2}\Big[\frac{g_1}{3}B\cdot S_R^d - 3\frac{g_3}{2}(S_R^d)^2\Big]. \tag{D.81}$$

The calculation of the terms corresponding to the u quark is totally resemblant, and the same sums are introduced:

$$S_L^u = D_L^5 + D_L^6 + D_L^7;\ S_R^u = D_R^5 + D_R^6 + D_R^7. \tag{D.82}$$

For the left waves we have:

$$S^3 = \frac{\mathrm{m}}{q_3}\Big[-\frac{g_1}{6}B\cdot S_L^u + \frac{g_2}{4}[(S_L^{25})^2 + (S_L^{36})^2 + (S_L^{47})^2] + \frac{3g_3}{2}(S_L^u)^2\Big]. \tag{D.83}$$

And we obtain for the right waves of the u quark:

$$S^4 = \frac{\mathrm{m}}{q_4}\Big[-\frac{2g_1}{3}B\cdot S_R^u - 3\frac{g_3}{2}(S_R^u)^2\Big]. \tag{D.84}$$

This gives:

$$\begin{aligned} S = {} & \frac{\mathrm{m}}{q_1}\Big[-\frac{g_1}{6}B\cdot S_L^d + \frac{g_2}{4}\big[(S_L^{25})^2 + (S_L^{36})^2 + (S_L^{47})^2\big] + \frac{3g_3}{2}(S_L^d)^2\Big] \\ & + \frac{\mathrm{m}}{q_2}\Big[\frac{g_1}{3}B\cdot S_R^d - \frac{3g_3}{2}(S_R^d)^2\Big] \\ & + \frac{\mathrm{m}}{q_3}\Big[-\frac{g_1}{6}B\cdot S_L^u + \frac{g_2}{4}\big[(S_L^{25})^2 + (S_L^{36})^2 + (S_L^{47})^2\big] + \frac{3g_3}{2}(S_L^u)^2\Big] \\ & + \frac{\mathrm{m}}{q_4}\Big[-\frac{2g_1}{3}B\cdot S_R^u - \frac{3g_3}{2}(S_R^u)^2\Big]. \end{aligned} \tag{D.85}$$

D.4 Calculation of $\Gamma^\rho_{\mu\nu}$

D.4.1 Calculation of $\mathcal{S}_{(k)}$ and $\mathcal{A}_{(k)}$

The S_k are bivectors in Cl_3. So they read:

$$\begin{aligned} \phi\sigma_k\overline{\phi} = S_k &:= \vec{E}_k + i\vec{H}_k, \\ \vec{E}_k &:= E_k^1\sigma_1 + E_k^2\sigma_2 + E_k^3\sigma_3, \\ \vec{H}_k &:= H_k^1\sigma_1 + H_k^2\sigma_2 + H_k^3\sigma_3. \end{aligned} \tag{D.86}$$

We thus have:

$$E_k^1 = S_k^{23}\ ;\ \ E_k^2 = S_k^{31}\ ;\ \ E_k^3 = S_k^{12}\ ;\ \ H_k^j = S_k^{j0}. \tag{D.87}$$

Next we obtain:

$$\begin{aligned}
\nabla S_k^\dagger &= (\partial_0 - \vec{\partial})(\vec{E}_k - i\vec{H}_k) \\
&= -\vec{\partial}\cdot\vec{E}_k + (\partial_0\vec{E}_k - \vec{\partial}\times\vec{H}_k) + i(-\vec{\partial}\times\vec{E}_k - \partial_0\vec{H}_k) + i\vec{\partial}\cdot\vec{H}_k \\
&= j_k + ij'_k = j_k^0 + \vec{j}_k + i\vec{j_k}' + i{j'}_k^0, \qquad \text{(D.88)}
\end{aligned}$$

with

$$j_k^0 := -\vec{\partial}\cdot\vec{E}_k \ ; \quad \vec{j}_k := \partial_0\vec{E}_k - \vec{\partial}\times\vec{H}_k, \qquad \text{(D.89)}$$

$${j'}_k^0 := \vec{\partial}\cdot\vec{H}_k \ ; \quad \vec{j_k}' := -\partial_0\vec{H}_k - \vec{\partial}\times\vec{E}_k. \qquad \text{(D.90)}$$

With the electromagnetic potential A we have:

$$\begin{aligned}
AS_k^\dagger &= (A^0 + \vec{A})(\vec{E}_k - i\vec{H}_k) \\
&= \vec{A}\cdot\vec{E}_k + (A^0\vec{E}_k + \vec{A}\times\vec{H}_k) + i(\vec{A}\times\vec{E}_k - A^0\vec{H}_k) - i\vec{A}\cdot\vec{H}_k \\
&= v_k + iv'_k = v_k^0 + \vec{v}_k + i\vec{v}_k' + i{v'}_k^0, \qquad \text{(D.91)}
\end{aligned}$$

with

$$v_k^0 := \vec{A}\cdot\vec{E}_k \ ; \quad \vec{v}_k := A^0\vec{E}_k + \vec{A}\times\vec{H}_k, \qquad \text{(D.92)}$$

$${v'}_k^0 := -\vec{A}\cdot\vec{H}_k \ ; \quad \vec{v}_k' := -A^0\vec{H}_k + \vec{A}\times\vec{E}_k. \qquad \text{(D.93)}$$

Similarly with v and S_k we have:

$$\begin{aligned}
JS_k^\dagger &= \rho \mathrm{v} S_k^\dagger = \rho(\mathrm{v}^0 + \vec{\mathrm{v}})(\vec{E}_k - i\vec{H}_k) \\
&= \rho[\vec{\mathrm{v}}\cdot\vec{E}_k + (\mathrm{v}^0\vec{E}_k + \vec{\mathrm{v}}\times\vec{H}_k) + i(\vec{\mathrm{v}}\times\vec{E}_k - \mathrm{v}^0\vec{H}_k) - i\vec{\mathrm{v}}\cdot\vec{H}_k] \\
&= \phi\phi^\dagger(\phi\sigma_k\overline{\phi})^\dagger = \phi\phi^\dagger\widehat{\phi}\sigma_k\phi^\dagger = \phi\rho e^{-i\beta}\sigma_k\phi^\dagger = e^{-i\beta}\phi\sigma_k\phi^\dagger = (\Omega_1 - i\Omega_2)\mathrm{D}_k \\
&= \Omega_1\mathrm{D}_k^0 + \Omega_1\vec{\mathrm{D}}_k - i\Omega_2\vec{\mathrm{D}}_k - i\Omega_2\mathrm{D}_k^0 \qquad \text{(D.94)}
\end{aligned}$$

Thus we obtain:

$$\frac{\Omega_1}{\rho}\mathrm{D}_k^0 = \vec{\mathrm{v}}\cdot\vec{E}_k \ ; \quad \frac{\Omega_1}{\rho}\vec{\mathrm{D}}_k = \mathrm{v}^0\vec{E}_k + \vec{\mathrm{v}}\times\vec{H}_k, \qquad \text{(D.95)}$$

$$\frac{\Omega_2}{\rho}\mathrm{D}_k^0 = \vec{\mathrm{v}}\cdot\vec{H}_k \ ; \quad \frac{\Omega_2}{\rho}\vec{\mathrm{D}}_k = \mathrm{v}^0\vec{H}_k - \vec{\mathrm{v}}\times\vec{E}_k. \qquad \text{(D.96)}$$

With the definition (4.29) we have:

$$\begin{aligned}
\mathcal{S}_{(k)} + i\mathcal{S}'_{(k)} &= \frac{\nabla S_k^\dagger}{\det(\phi^\dagger)} = \frac{\Omega_1 + i\Omega_2}{\rho^2}(j_k + ij'_k) \\
&= \rho^{-2}[\Omega_1 j_k - \Omega_2 j'_k + i(\Omega_1 j'_k + \Omega_2 j_k)], \qquad \text{(D.97)}
\end{aligned}$$

$$\rho^2\mathcal{S}_{(k)} = \Omega_1 j_k - \Omega_2 j'_k, \qquad \text{(D.98)}$$

$$\rho^2\mathcal{S}'_{(k)} = \Omega_1 j'_k + \Omega_2 j_k. \qquad \text{(D.99)}$$

Similarly we have :

$$\mathcal{A}_{(k)} + i\mathcal{A}'_{(k)} = \frac{AS_k^\dagger}{\det(\phi^\dagger)} = \frac{\Omega_1 + i\Omega_2}{\rho^2}(v_k + iv'_k)$$
$$= \rho^{-2}[\Omega_1 v_k - \Omega_2 v'_k + i(\Omega_1 v'_k + \Omega_2 v_k)], \tag{D.100}$$
$$\rho^2\mathcal{A}_{(k)} = \Omega_1 v_k - \Omega_2 v'_k, \tag{D.101}$$
$$\rho^2\mathcal{A}'_{(k)} = \Omega_1 v'_k + \Omega_2 v_k. \tag{D.102}$$

D.4.2 Calculation of $\Gamma^\mu_{\mu\nu}$

We start from the definition of Durand's spin density (4.31), which gives:

$$\tau = \frac{1}{2}[(\nabla\widehat{\phi})\phi^\dagger - \sigma^\mu\widehat{\phi}\partial_\mu\phi^\dagger],$$
$$2\tau = (\nabla\widehat{\phi})\phi^\dagger - \dot{\nabla}\widehat{\phi}\dot{\phi}^\dagger, \tag{D.103}$$

where the dots indicate that which we derive. And we have:

$$\nabla(\widehat{\phi}\phi^\dagger) = (\nabla\widehat{\phi})\phi^\dagger + \dot{\nabla}\widehat{\phi}\dot{\phi}^\dagger. \tag{D.104}$$

Hence by adding we get:

$$2\tau + \nabla(\Omega_1 - i\Omega_2) = 2(\nabla\widehat{\phi})\phi^\dagger. \tag{D.105}$$

With our improved wave equation we have:

$$(\nabla\widehat{\phi})\phi^\dagger = qA\widehat{\phi}\sigma_{21}\phi^\dagger + e^{-i\beta}\phi\mathbf{m}\sigma_{21}\phi^\dagger, \tag{D.106}$$
$$\phi\mathbf{m}\sigma_{21}\phi^\dagger = -i(\mathrm{lD}_R - \mathbf{r}\mathrm{D}_L), \tag{D.107}$$
$$(\nabla\widehat{\phi})\phi^\dagger = -iqAS_3^\dagger - i\Big(\frac{\Omega_1}{\rho} - i\frac{\Omega_2}{\rho}\Big)(\mathrm{lD}_R - \mathbf{r}\mathrm{D}_L). \tag{D.108}$$

We now let:

$$\tau = \tau_1 + i\tau_2 \ ; \quad \tau_1 = \frac{1}{2}(\tau + \tau^\dagger) \ ; \quad i\tau_2 = \frac{1}{2}(\tau - \tau^\dagger). \tag{D.109}$$

With (D.105) and (D.108) we obtain:

$$\nabla\Omega_1 - i\nabla\Omega_2 = -2(\tau_1 + i\tau_2) - 2iqAS_3^\dagger - 2\Big(\frac{\Omega_2}{\rho} + i\frac{\Omega_1}{\rho}\Big)(\mathrm{lD}_R - \mathbf{r}\mathrm{D}_L)$$
$$= -2(\tau_1 + i\tau_2) - 2iq(v_3 + iv'_3) - 2\Big(\frac{\Omega_2}{\rho} + i\frac{\Omega_1}{\rho}\Big)(\mathrm{lD}_R - \mathbf{r}\mathrm{D}_L). \tag{D.110}$$

This gives:

$$\nabla\Omega_1 = -2\tau_1 + 2qv'_3 - 2\frac{\Omega_2}{\rho}(\mathrm{lD}_R - \mathbf{r}\mathrm{D}_L), \tag{D.111}$$
$$\nabla\Omega_2 = 2\tau_2 + 2qv_3 + 2\frac{\Omega_1}{\rho}(\mathrm{lD}_R - \mathbf{r}\mathrm{D}_L). \tag{D.112}$$

Since $\rho^2 = \Omega_1^2 + \Omega_2^2$, we have:

$$\rho\nabla\rho = \Omega_1\nabla\Omega_1 + \Omega_2\nabla\Omega_2. \tag{D.113}$$

With our improved wave equation and with (D.111) and (D.112) we get:

$$\begin{aligned}\rho\nabla\rho &= \Omega_1[-2\tau_1 + 2qv_3' - 2\frac{\Omega_2}{\rho}(\mathbf{l}\mathrm{D}_R - \mathbf{r}\mathrm{D}_L)] \\ &\quad + \Omega_2[2\tau_2 + 2qv_3 + 2m\frac{\Omega_1}{\rho}(\mathbf{l}\mathrm{D}_R - \mathbf{r}\mathrm{D}_L)] \\ &= 2(-\Omega_1\tau_1 + \Omega_2\tau_2) + 2q(\Omega_1 v_3' + \Omega_2 v_3). \end{aligned} \tag{D.114}$$

And with (D.102) we have:

$$\rho\nabla\rho = 2(-\Omega_1\tau_1 + \Omega_2\tau_2) + 2q\rho^2\mathcal{A}'_{(3)}. \tag{D.115}$$

With (4.32) we get:

$$\rho^2(\mathcal{T}+i\mathcal{T}') = (\tau_1+i\tau_2)(\Omega_1+i\Omega_2) = (\tau_1\Omega_1-\tau_2\Omega_2)+i(\tau_1\Omega_2+\tau_2\Omega_1). \tag{D.116}$$

Dividing (D.115) by ρ^2 we finally have:

$$\nabla(\ln\rho) = -2\mathcal{T} + 2q\mathcal{A}'_{(3)}. \tag{D.117}$$

With (4.25) we have:

$$\Gamma^0_{0\nu} = D^\mu_\nu\partial_\mu(\ln\rho) = D_\nu\cdot\nabla(\ln\rho) \tag{D.118}$$

$$= D_\nu\cdot(-2\mathcal{T} + 2q\mathcal{A}'_{(3)}), \tag{D.119}$$

which is (4.39).

D.4.3 Calculation of $\Gamma^0_{j\nu}$ and $\Gamma^j_{0\nu}$, $j = 1,\ 2,\ 3$

We start from:

$$\begin{aligned}\Gamma^0_{j\nu} &= \rho^{-2}(\boldsymbol{\partial}_\nu D^\mu_j)\overline{D}^0_\mu = \rho^{-2}[(\boldsymbol{\partial}_\nu D^0_j)\overline{D}^0_0 + \sum_{k=1}^{3}(\boldsymbol{\partial}_\nu D^k_j)\overline{D}^0_k] \\ &= \rho^{-2}[(\boldsymbol{\partial}_\nu D^0_j)D^0_0 - \sum_{k=1}^{3}(\boldsymbol{\partial}_\nu D^k_j)D^k_0],\end{aligned} \tag{D.120}$$

and similarly:

$$\begin{aligned}\Gamma^j_{0\nu} &= \rho^{-2}(\boldsymbol{\partial}_\nu D^\mu_0)\overline{D}^j_\mu = \rho^{-2}[(\boldsymbol{\partial}_\nu D^0_0)\overline{D}^j_0 + \sum_{k=1}^{3}(\boldsymbol{\partial}_\nu D^k_0)\overline{D}^j_k] \\ &= \rho^{-2}[-(\boldsymbol{\partial}_\nu D^0_0)D^0_j + \sum_{k=1}^{3}(\boldsymbol{\partial}_\nu D^k_0)D^k_j].\end{aligned} \tag{D.121}$$

We thus have:

$$\Gamma^0_{j\nu} - \Gamma^j_{0\nu} = \rho^{-2}[\boldsymbol{\partial}_\nu(D^0_0 D^0_j) - \sum_{k=1}^{3}\boldsymbol{\partial}_\nu(D^k_0 D^k_j)] = \rho^{-2}\boldsymbol{\partial}_\nu(D_0 \cdot D_j) = 0,$$

$$\Gamma^0_{j\nu} = \Gamma^j_{0\nu}. \tag{D.122}$$

We also get:

$$\begin{aligned}\Gamma^0_{j\nu} &= \rho^{-2}\begin{pmatrix} D^0_0(D^0_\nu\partial_0 + D^1_\nu\partial_1 + D^2_\nu\partial_2 + D^3_\nu\partial_3)(D^0_j) \\ -D^1_0(D^0_\nu\partial_0 + D^1_\nu\partial_1 + D^2_\nu\partial_2 + D^3_\nu\partial_3)(D^1_j) \\ -D^2_0(D^0_\nu\partial_0 + D^1_\nu\partial_1 + D^2_\nu\partial_2 + D^3_\nu\partial_3)(D^2_j) \\ -D^0_0(D^0_\nu\partial_0 + D^1_\nu\partial_1 + D^2_\nu\partial_2 + D^3_\nu\partial_3)(D^3_j) \end{pmatrix} \\ &= \rho^{-2}\begin{pmatrix} D^0_\nu(D^0_0\partial_0 D^0_j - D^1_0\partial_0 D^1_j - D^2_0\partial_0 D^2_j - D^3_0\partial_0 D^3_j) \\ +D^1_\nu(D^0_0\partial_1 D^0_j - D^1_0\partial_1 D^1_j - D^2_0\partial_1 D^2_j - D^3_0\partial_1 D^3_j) \\ +D^2_\nu(D^0_0\partial_2 D^0_j - D^1_0\partial_2 D^1_j - D^2_0\partial_2 D^2_j - D^3_0\partial_2 D^3_j) \\ +D^3_\nu(D^0_0\partial_3 D^0_j - D^1_0\partial_3 D^1_j - D^2_0\partial_3 D^2_j - D^3_0\partial_3 D^3_j) \end{pmatrix} \\ &= \rho^{-2}D^\mu_\nu(D^0_0\partial_\mu D^0_j - D^1_0\partial_\mu D^1_j - D^2_0\partial_\mu D^2_j - D^3_0\partial_\mu D^3_j). \end{aligned} \tag{D.123}$$

Next we obtain:

$$\begin{aligned}\Gamma^0_{j\nu} &= \rho^{-2}D^\mu_\nu[\partial_\mu(D^0_0 D^0_j) - \partial_\mu(D^1_0 D^1_j) - \partial_\mu(D^2_0 D^2_j) - \partial_\mu(D^3_0 D^3_j)] \\ &\quad + \rho^{-2}D^\mu_\nu[-D^0_j\partial_\mu D^0_0 + D^1_j\partial_\mu D^1_0 + D^2_j\partial_\mu D^2_0 + D^3_j\partial_\mu D^3_0] \end{aligned} \tag{D.124}$$

$$\begin{aligned} &= \rho^{-2}D^\mu_\nu[\partial_\mu(D_0 \cdot D_j) - D^0_j\partial_\mu D^0_0 + D^1_j\partial_\mu D^1_0 + D^2_j\partial_\mu D^2_0 + D^3_j\partial_\mu D^3_0] \\ &= \rho^{-2}D^\mu_\nu[-D^0_j\partial_\mu D^0_0 + D^1_j\partial_\mu D^1_0 + D^2_j\partial_\mu D^2_0 + D^3_j\partial_\mu D^3_0]. \end{aligned} \tag{D.125}$$

We let:

$$\Gamma^0_{j\nu} = \rho^{-2}D^\mu_\nu X_{j\mu}, \tag{D.126}$$

$$X_{j\mu} = -D^0_j\partial_\mu D^0_0 + D^1_j\partial_\mu D^1_0 + D^2_j\partial_\mu D^2_0 + D^3_j\partial_\mu D^3_0. \tag{D.127}$$

D.4.4 Calculation of $\Gamma^0_{1\nu}$

We start from:

$$\Gamma^0_{1\nu} = \rho^{-2}D^\mu_\nu X_{1\mu}, \tag{D.128}$$

$$X_{1\mu} = -D^0_1\partial_\mu D^0_0 + D^1_1\partial_\mu D^1_0 + D^2_1\partial_\mu D^2_0 + D^3_1\partial_\mu D^3_0. \tag{D.129}$$

We have:

$$\begin{aligned} X_{1\mu} &= -(-\xi^*_1\eta^*_2 - \xi_1\eta_2 + \xi^*_2\eta^*_1 + \xi_2\eta_1)\partial_\mu(\xi_1\xi^*_1 + \xi_2\xi^*_2 + \eta_1\eta^*_1 + \eta_2\eta^*_2) \\ &\quad + (\xi^*_1\eta^*_1 - \xi_2\eta_2 - \xi^*_2\eta^*_2 + \xi_1\eta_1)\partial_\mu(\xi_1\xi^*_2 + \xi_2\xi^*_1 - \eta_1\eta^*_2 - \eta_2\eta^*_1) \\ &\quad + i(-\xi^*_1\eta^*_1 + \xi_2\eta_2 - \xi^*_2\eta^*_2 + \xi_1\eta_1)\partial_\mu i(\xi_1\xi^*_2 - \xi_2\xi^*_1 - \eta_1\eta^*_2 + \eta_2\eta^*_1) \\ &\quad + (-\xi^*_1\eta^*_2 - \xi_1\eta_2 - \xi^*_2\eta^*_1 - \xi_2\eta_1)\partial_\mu(\xi_1\xi^*_1 - \xi_2\xi^*_2 - \eta_1\eta^*_1 + \eta_2\eta^*_2). \end{aligned} \tag{D.130}$$

This gives:

$$X_{1\mu} = 2\begin{pmatrix} -\xi_2(\eta_1\xi_1^* + \eta_2\xi_2^*)\partial_\mu\xi_1 + \xi_1(\eta_1\xi_1^* + \eta_2\xi_2^*)\partial_\mu\xi_2 \\ +\eta_2(\xi_1\eta_1^* + \xi_2\eta_2^*)\partial_\mu\eta_1 - \eta_1(\xi_1\eta_1^* + \xi_2\eta_2^*)\partial_\mu\eta_2 \\ -\xi_2^*(\xi_1\eta_1^* + \xi_2\eta_2^*)\partial_\mu\xi_1^* + \xi_1^*(\xi_1\eta_1^* + \xi_2\eta_2^*)\partial_\mu\xi_2^* \\ +\eta_2^*(\eta_1\xi_1^* + \eta_2\xi_2^*)\partial_\mu\eta_1^* - \eta_1^*(\eta_1\xi_1^* + \eta_2\xi_2^*)\partial_\mu\eta_2^* \end{pmatrix}, \qquad (D.131)$$

and with (A.84) we get:

$$X_{1\mu} = \begin{pmatrix} -\xi_2(\Omega_1 - i\Omega_2)\partial_\mu\xi_1 + \xi_1(\Omega_1 - i\Omega_2)\partial_\mu\xi_2 \\ +\eta_2(\Omega_1 + i\Omega_2)\partial_\mu\eta_1 - \eta_1(\Omega_1 + i\Omega_2)\partial_\mu\eta_2 \\ -\xi_2^*(\Omega_1 + i\Omega_2)\partial_\mu\xi_1^* + \xi_1^*(\Omega_1 + i\Omega_2)\partial_\mu\xi_2^* \\ +\eta_2^*(\Omega_1 - i\Omega_2)\partial_\mu\eta_1^* - \eta_1^*(\Omega_1 - i\Omega_2)\partial_\mu\eta_2^* \end{pmatrix}. \qquad (D.132)$$

We can thus express it as follows:

$$\begin{aligned} X_{1\mu} &= \Omega_1 Y_\mu + i\Omega_2 Z_\mu, && (D.133)\\ Y_\mu &= \xi_1\partial_\mu\xi_2 - \xi_2\partial_\mu\xi_1 - \eta_1\partial_\mu\eta_2 + \eta_2\partial_\mu\eta_1 \\ &\quad + \xi_1^*\partial_\mu\xi_2^* - \xi_2^*\partial_\mu\xi_1^* - \eta_1^*\partial_\mu\eta_2^* + \eta_2^*\partial_\mu\eta_1^*, && (D.134)\\ Z_\mu &= -\xi_1\partial_\mu\xi_2 + \xi_2\partial_\mu\xi_1 - \eta_1\partial_\mu\eta_2 + \eta_2\partial_\mu\eta_1 \\ &\quad + \xi_1^*\partial_\mu\xi_2^* - \xi_2^*\partial_\mu\xi_1^* + \eta_1^*\partial_\mu\eta_2^* - \eta_2^*\partial_\mu\eta_1^*. && (D.135) \end{aligned}$$

Our improved wave equation is equivalent to the system:

$$\begin{aligned} 0 &= (\nabla + ia)\eta;\ a := qA + \mathbf{l}\mathrm{v}, \\ 0 &= (\widehat{\nabla} + i\widehat{b})\xi;\ b := qA + \mathbf{r}\mathrm{v}. && (D.136) \end{aligned}$$

This is equivalent to the system of partial differential equations:

$$\begin{aligned} 0 &= \partial_0\eta_1 - \partial_1\eta_2 + i\partial_2\eta_2 - \partial_3\eta_1 + i(a_0\eta_1 - a_1\eta_2 + ia_2\eta_2 - a_3\eta_1), \\ 0 &= \partial_0\eta_2 - \partial_1\eta_1 - i\partial_2\eta_1 + \partial_3\eta_2 + i(a_0\eta_2 - a_1\eta_1 - ia_2\eta_1 + a_3\eta_2), \quad (D.137) \\ 0 &= \partial_0\xi_1 + \partial_1\xi_2 - i\partial_2\xi_2 + \partial_3\xi_1 + i(b_0\xi_1 + b_1\xi_2 - ib_2\xi_2 + b_3\xi_1), \\ 0 &= \partial_0\xi_2 + \partial_1\xi_1 + i\partial_2\xi_1 - \partial_3\xi_2 + i(b_0\xi_2 + b_1\xi_1 + ib_2\xi_1 - b_3\xi_2). \end{aligned}$$

Given these systems of equations, after simplification we get:

$$\begin{aligned} Y_0 &= -\partial_1 S_1^{23} - \partial_2 S_1^{31} - \partial_3 S_1^{12} \\ &\quad + 2b_1 H_R^1 + 2b_2 H_R^2 + 2b_3 H_R^3 - 2a_1 H_L^1 - 2a_2 H_L^2 - 2a_3 H_L^3 \\ &= -\vec{\partial}\cdot\vec{E}_1 + 2(q\vec{A} + \mathbf{r}\vec{\mathrm{v}})\cdot\vec{H}_R - 2(q\vec{A} + \mathbf{l}\vec{\mathrm{v}})\cdot\vec{H}_L \\ &= -\vec{\partial}\cdot\vec{E}_1 + 2q\vec{A}\cdot\vec{E}_2 + 2m\vec{\mathrm{v}}\cdot\vec{E}_2 - 2d\vec{\mathrm{v}}\cdot\vec{H}_1 \\ &= j_1^0 + 2qv_2^0 + \frac{2m}{\rho}\Omega_1\mathrm{D}_2^0 - \frac{2d}{\rho}\Omega_2\mathrm{D}_1^0. && (D.138) \end{aligned}$$

Similarly we have:

$$\begin{aligned}
Z_0 &= i\partial_1 S_1^{10} + i\partial_2 S_1^{20} + i\partial_3 S_1^{30} \\
&+ 2i(b_1 E_R^1 + b_2 E_R^2 + b_3 E_R^3) - 2i(a_1 E_L^1 + a_2 E_L^2 + a_3 E_L^3) \\
&= i(\vec{\partial} \cdot \vec{H}_1 + 2\vec{b} \cdot \vec{E}_R - 2\vec{a} \cdot \vec{E}_L) \qquad (D.139) \\
&= i[\vec{\partial} \cdot \vec{H}_1 - 2q\vec{A} \cdot \vec{H}_2 - 2m\vec{\mathrm{v}} \cdot \vec{H}_2 - 2d\vec{\mathrm{v}} \cdot \vec{E}_1]. \\
&= i[{j'}_1^0 + 2q{v'}_2^0 - \frac{2m}{\rho}\Omega_2 \mathrm{D}_2^0 - \frac{2d}{\rho}\Omega_1 \mathrm{D}_1^0]. \qquad (D.140)
\end{aligned}$$

We hence obtain:

$$\begin{aligned}
X_{10} &= \Omega_1 Y_0 + i\Omega_2 Z_0 \\
&= \Omega_1(j_1^0 - 2qv_2^0 - 2m\frac{\Omega_1}{\rho}D_2^0 + \frac{2d}{\rho}\Omega_2 \mathrm{D}_1^0) + i\Omega_2(i{j'}_1^0 - 2iq{v'}_2^0 \\
&+ 2im\frac{\Omega_2}{\rho}D_2^0 - \frac{2d}{\rho}\Omega_1 \mathrm{D}_1^0) \\
&= \Omega_1 j_1^0 - \Omega_2 {j'}_1^0 + 2q(-\Omega_1 v_2^0 + \Omega_2 {v'}_2^0) - 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho}D_2^0 \\
&= \rho^2(\mathcal{S}_{(1)}^0 - 2q\mathcal{A}_{(2)}^0) - 2m\rho D_2^0.
\end{aligned}$$

Again using (D.137) and simplifying, we get:

$$\begin{aligned}
Y_1 &= -\partial_0 S_1^{23} + \partial_2 S_1^{30} - \partial_3 S_1^{20} + 2q(A_0 E_2^1 - A_2 H_2^3 + A_3 H_2^2) \\
&+ 2m(\mathrm{v}_0 E_2^1 - \mathrm{v}_2 H_2^3 + \mathrm{v}_3 H_2^2) - 2d(\mathrm{v}_0 H_1^1 + \mathrm{v}_2 E_1^3 - \mathrm{v}_3 E_1^2) \\
&= -j_1^1 + 2qv_2^1 + 2m\frac{\Omega_1}{\rho}\mathrm{D}_2^1 - 2d\frac{\Omega_2}{\rho}\mathrm{D}_1^1. \qquad (D.141)
\end{aligned}$$

Similarly, after simplification (D.137) gives:

$$\begin{aligned}
Z_1 &= i[\partial_0 H_1^1 + \partial_2 E_1^3 - \partial_3 E_1^2 - 2q(A^0 H_2^1 - A^3 E_2^2 + A^2 E_2^3) \\
&- 2m(\mathrm{v}_0 H_2^1 - \mathrm{v}^2 E_2^3 + \mathrm{v}^3 E_2^2) - 2d(\mathrm{v}_0 E_1^1 + \mathrm{v}^2 H_1^3 - \mathrm{v}^3 H_1^2) \\
&= i\Big[- {j'}_1^1 + 2q{v'}_2^1 - 2m\frac{\Omega_2}{\rho}\mathrm{D}_2^1 - 2d\frac{\Omega_1}{\rho}\mathrm{D}_1^1\Big]. \qquad (D.142)
\end{aligned}$$

With (D.133) and (D.137) we have:

$$\begin{aligned}
X_{11} &= \Omega_1 Y_1 + i\Omega_2 Z_1 \\
&= \Omega_1(-j_1^1 + 2qv_2^1 + 2m\frac{\Omega_1}{\rho}D_2^1 - 2d\frac{\Omega_2}{\rho}\mathrm{D}_1^1) \\
&- \Omega_2(-{j'}_1^1 + 2q{v'}_2^1 - 2m\frac{\Omega_2}{\rho}\mathrm{D}_2^1 - 2d\frac{\Omega_1}{\rho}\mathrm{D}_1^1) \\
&= -\Omega_1 j_1^1 + \Omega_2 {j'}_1^1 + 2q(\Omega_1 v_2^1 - \Omega_2 {v'}_2^1) + 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho}D_2^1 \\
&= \rho^2(-\mathcal{S}_{(1)}^1 + 2q\mathcal{A}_{(2)}^1) + 2m\rho D_2^1. \qquad (D.143)
\end{aligned}$$

Again using (D.137) and simplifying we have:

$$\begin{aligned}Y_2 &= -\partial_0 S_1^{31} + \partial_3 S_1^{10} - \partial_1 S_1^{30} + 2q(A_0 E_2^2 - A_3 H_2^1 + A_1 H_2^3)\\ &+ 2m(\mathrm{v}_0 E_2^2 - \mathrm{v}_3 H_2^1 + \mathrm{v}_1 H_2^3) - 2d(\mathrm{v}_0 H_1^2 + \mathrm{v}_3 E_1^1 - \mathrm{v}_1 E_1^3)\\ &= -j_1^2 + 2qv_2^2 + 2m\frac{\Omega_1}{\rho}\mathrm{D}_2^2 - 2d\frac{\Omega_2}{\rho}\mathrm{D}_1^2. \end{aligned} \tag{D.144}$$

Similarly, after simplification (D.137) gives:

$$\begin{aligned}Z_2 &= i[\partial_0 H_1^2 + \partial_3 E_1^1 - \partial_1 E_1^3 - 2q(A^0 H_2^2 - A^1 E_2^3 + A^3 E_2^1)\\ &- 2m(\mathrm{v}_0 H_2^2 - \mathrm{v}^3 E_2^1 + \mathrm{v}^1 E_2^3) - 2d(\mathrm{v}_0 E_1^2 + \mathrm{v}^3 H_1^1 - \mathrm{v}^1 H_1^3)\\ &= i[-{j'}_1^2 + 2q{v'}_2^2 - 2m\frac{\Omega_2}{\rho}\mathrm{D}_2^2 - 2d\frac{\Omega_1}{\rho}\mathrm{D}_1^2]. \end{aligned} \tag{D.145}$$

With (D.133) and (D.137) we have:

$$\begin{aligned}X_{12} &= \Omega_1 Y_2 + i\Omega_2 Z_2\\ &= \Omega_1(-j_1^2 + 2qv_2^2 + 2m\frac{\Omega_1}{\rho}D_2^2 - 2d\frac{\Omega_2}{\rho}\mathrm{D}_1^2)\\ &- \Omega_2(-{j'}_1^2 + 2q{v'}_2^2 - 2m\frac{\Omega_2}{\rho}\mathrm{D}_2^2 - 2d\frac{\Omega_1}{\rho}\mathrm{D}_1^2)\\ &= -\Omega_1 j_1^2 + \Omega_2 {j'}_1^2 + 2q(\Omega_1 v_2^2 - \Omega_2 {v'}_2^2) + 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho}D_2^2\\ &= \rho^2(-\mathcal{S}_{(1)}^2 + 2q\mathcal{A}_{(2)}^2) + 2m\rho D_2^2. \end{aligned} \tag{D.146}$$

Again using (D.137), and simplifying we have:

$$\begin{aligned}Y_3 &= -\partial_0 S_1^{12} + \partial_1 S_1^{10} - \partial_2 S_1^{10} + 2q(A_0 E_2^3 - A_1 H_2^2 + A_2 H_2^1)\\ &+ 2m(\mathrm{v}_0 E_2^3 - \mathrm{v}_1 H_2^2 + \mathrm{v}_2 H_2^1) - 2d(\mathrm{v}_0 H_1^3 + \mathrm{v}_1 E_1^2 - \mathrm{v}_2 E_1^1)\\ &= -j_1^3 + 2qv_2^3 + 2m\frac{\Omega_1}{\rho}\mathrm{D}_2^3 - 2d\frac{\Omega_2}{\rho}\mathrm{D}_1^3. \end{aligned} \tag{D.147}$$

Similarly, and after simplification, (D.137) gives:

$$\begin{aligned}Z_3 &= i[\partial_0 H_1^3 + \partial_1 E_1^2 - \partial_2 E_1^1 - 2q(A^0 H_2^3 - A^2 E_2^1 + A^1 E_2^2)\\ &- 2m(\mathrm{v}_0 H_2^3 - \mathrm{v}^1 E_2^2 + \mathrm{v}^2 E_2^1) - 2d(\mathrm{v}_0 E_1^3 + \mathrm{v}^1 H_1^2 - \mathrm{v}^2 H_1^1)\\ &= i[-{j'}_1^3 + 2q{v'}_2^3 - 2m\frac{\Omega_2}{\rho}\mathrm{D}_2^3 - 2d\frac{\Omega_1}{\rho}\mathrm{D}_1^3]. \end{aligned} \tag{D.148}$$

With (D.133) and (D.137) we have:

$$
\begin{aligned}
X_{13} &= \Omega_1 Y_3 + i\Omega_2 Z_3 \\
&= \Omega_1(-j_1^3 + 2qv_2^3 + 2m\frac{\Omega_1}{\rho}D_2^3 - 2d\frac{\Omega_2}{\rho}\mathrm{D}_1^3) \\
&- \Omega_2(-{j'}_1^3 + 2q{v'}_2^3 - 2m\frac{\Omega_2}{\rho}\mathrm{D}_2^3 - 2d\frac{\Omega_1}{\rho}\mathrm{D}_1^3) \\
&= -\Omega_1 j_1^3 + \Omega_2 {j'}_1^3 + 2q(\Omega_1 v_2^3 - \Omega_2 {v'}_2^3) + 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho}D_2^3 \\
&= \rho^2(-\mathcal{S}_{(1)}^3 + 2q\mathcal{A}_{(2)}^3) + 2m\rho D_2^3.
\end{aligned} \tag{D.149}
$$

And we thus get:

$$
\begin{aligned}
\Gamma_{1\nu}^0 &= \rho^{-2}D_\nu^\mu X_{1\mu} = \rho^{-2}[D_\nu^0 X_{10} + \sum_{k=1}^3 D_\nu^k X_{1k}] \\
&= \rho^{-2}[D_\nu^0(\rho^2\mathcal{S}_{(1)}^0 - 2q\rho^2\mathcal{A}_{(2)}^0 - 2m\rho D_2^0) \\
&+ \sum_{k=1}^3 D_\nu^k(-\rho^2\mathcal{S}_{(1)}^k + 2q\rho^2\mathcal{A}_{(2)}^k + 2m\rho D_2^k)] \\
&= D_\nu \cdot (\mathcal{S}_{(1)} - 2q\mathcal{A}_{(2)} - 2\frac{m}{\rho}D_2) \\
&= D_\nu \cdot (\mathcal{S}_{(1)} - 2q\mathcal{A}_{(2)}) + 2m\rho\delta_\nu^2,
\end{aligned} \tag{D.150}
$$

which is (4.33)

D.4.5 Calculation of $\Gamma_{2\nu}^0$

We start from:

$$\Gamma_{2\nu}^0 = \rho^{-2}D_\nu^\mu X_{2\mu}, \tag{D.151}$$

$$X_{2\mu} = -D_2^0\partial_\mu D_0^0 + D_2^1\partial_\mu D_0^1 + D_2^2\partial_\mu D_0^2 + D_2^3\partial_\mu D_0^3. \tag{D.152}$$

We have:

$$
\begin{aligned}
X_{2\mu} = &-i(-\xi_1^*\eta_2^* + \xi_1\eta_2 + \xi_2^*\eta_1^* - \xi_2\eta_1)\partial_\mu(\xi_1\xi_1^* + \xi_2\xi_2^* + \eta_1\eta_1^* + \eta_2\eta_2^*) \\
&+ i(\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* - \xi_1\eta_1)\partial_\mu(\xi_1\xi_2^* + \xi_2\xi_1^* - \eta_1\eta_2^* - \eta_2\eta_1^*) \\
&+ (\xi_1^*\eta_1^* + \xi_2\eta_2 + \xi_2^*\eta_2^* + \xi_1\eta_1)\partial_\mu i(\xi_1\xi_2^* - \xi_2\xi_1^* - \eta_1\eta_2^* + \eta_2\eta_1^*) \\
&+ i(-\xi_1^*\eta_2^* + \xi_1\eta_2 - \xi_2^*\eta_1^* + \xi_2\eta_1)\partial_\mu(\xi_1\xi_1^* - \xi_2\xi_2^* - \eta_1\eta_1^* + \eta_2\eta_2^*),
\end{aligned} \tag{D.153}
$$

which with (A.84) gives:

$$
X_{2\mu} = \begin{pmatrix}
\xi_2(i\Omega_1 + \Omega_2)\partial_\mu\xi_1 - \xi_1(i\Omega_1 + \Omega_2)\partial_\mu\xi_2 \\
-\eta_2(i\Omega_1 - \Omega_2)\partial_\mu\eta_1 + \eta_1(i\Omega_1 - \Omega_2)\partial_\mu\eta_2 \\
-\xi_2^*(i\Omega_1 - \Omega_2)\partial_\mu\xi_1^* + \xi_1^*(i\Omega_1 - \Omega_2)\partial_\mu\xi_2^* \\
+\eta_2^*(i\Omega_1 + \Omega_2)\partial_\mu\eta_1^* - \eta_1^*(i\Omega_1 + \Omega_2)\partial_\mu\eta_2^*
\end{pmatrix}. \tag{D.154}
$$

We may then express it as:

$$X_{2\mu} = i\Omega_1 L_\mu + \Omega_2 M_\mu, \tag{D.155}$$

$$\begin{aligned} L_\mu &= -\xi_1\partial_\mu\xi_2 + \xi_2\partial_\mu\xi_1 + \eta_1\partial_\mu\eta_2 - \eta_2\partial_\mu\eta_1 \\ &\quad + \xi_1^*\partial_\mu\xi_2^* - \xi_2^*\partial_\mu\xi_1^* - \eta_1^*\partial_\mu\eta_2^* + \eta_2^*\partial_\mu\eta_1^*, \end{aligned} \tag{D.156}$$

$$\begin{aligned} M_\mu &= -\xi_1\partial_\mu\xi_2 + \xi_2\partial_\mu\xi_1 - \eta_1\partial_\mu\eta_2 + \eta_2\partial_\mu\eta_1 \\ &\quad - \xi_1^*\partial_\mu\xi_2^* + \xi_2^*\partial_\mu\xi_1^* - \eta_1^*\partial_\mu\eta_2^* + \eta_2^*\partial_\mu\eta_1^*. \end{aligned} \tag{D.157}$$

Again with (D.137) and after simplification we get:

$$\begin{aligned} L_0 &= i[\partial_1 E_2^1 + \partial_2 E_2^2 + \partial_3 E_2^3 - 2\vec{b}\cdot\vec{E}_R - 2\vec{a}\cdot\vec{E}_L \\ &= i[\vec{\partial}\cdot\vec{E}_2 - 2(q\vec{A} + \mathbf{r}\vec{\mathrm{v}})\cdot\vec{E}_R - 2(q\vec{A} + \mathbf{l}\vec{\mathrm{v}})\cdot\vec{E}_L] \\ &= i(-j_2^0 - 2qv_1^0 - 2m\frac{\Omega_1}{\rho}D_1^0 - 2d\frac{\Omega_2}{\rho}D_2^0). \end{aligned} \tag{D.158}$$

After simplification the equation (D.157) yields:

$$\begin{aligned} M_0 &= -\partial_1 H_2^1 - \partial_2 H_2^2 - \partial_3 H_2^3 + 2\vec{b}\cdot\vec{H}_R + 2\vec{a}\cdot\vec{H}_L \\ &= -\vec{\partial}\cdot\vec{H}_2 + 2(q\vec{A} + \mathbf{r}\vec{\mathrm{v}})\cdot\vec{H}_R + 2(q\vec{A} + \mathbf{l}\vec{\mathrm{v}})\cdot\vec{H}_L] \\ &= -{j'}_2^0 - 2q{v'}_1^0 + 2m\frac{\Omega_2}{\rho}D_1^0 - 2d\frac{\Omega_1}{\rho}D_2^0). \end{aligned} \tag{D.159}$$

With (D.155) and (D.158) we obtain:

$$\begin{aligned} X_{20} &= i\Omega_1 L_0 + \Omega_2 M_0 \\ &= -\Omega_1(-j_2^0 - 2qv_1^0 - 2m\frac{\Omega_1}{\rho}D_1^0 - 2d\frac{\Omega_2}{\rho}D_2^0) \\ &\quad + \Omega_2(-{j'}_2^0 - 2q{v'}_1^0 + 2m\frac{\Omega_2}{\rho}D_1^0 - 2d\frac{\Omega_1}{\rho}D_2^0) \\ &= (\Omega_1 j_2^0 - \Omega_2 {j'}_2^0) + 2q(\Omega_1 v_1^0 - \Omega_2 {v'}_1^0) + 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho}D_1^0 \\ &= \rho^2(\mathcal{S}^0_{(2)} + 2q\mathcal{A}^0_{(1)}) + 2m\rho D_1^0. \end{aligned} \tag{D.160}$$

As usual with (D.137) and after simplification we have:

$$\begin{aligned} L_1 &= i[\partial_0(H_R^1 - H_L^1) + \partial_2(E_R^3 - E_L^3) - \partial_3(E_R^2 - E_L^2) \\ &\quad + 2b_0E_R^1 + 2a_0E_L^1 - 2b_2H_R^3 - 2a_2H_L^3 + 2b_3H_R^2 + 2a_3H_L^2] \\ &= i\Big(j_2^1 + 2qv_1^1 + 2m\frac{\Omega_1}{\rho}D_1^1 + 2d\frac{\Omega_2}{\rho}D_2^1\Big). \end{aligned} \tag{D.161}$$

With (D.137) and after simplification we get:

$$\begin{aligned} M_1 &= \partial_0(E_R^1 - E_L^1) + \partial_2(-H_R^3 + H_L^3) + \partial_3(H_R^2 - H_L^2) \\ &\quad + 2(-b_0H_R^1 - b_2E_R^3 + b_3E_R^2) + 2(-a_0H_L^1 - a_2E_L^3 + a_3E_L^2) \\ &= {j'}_2^1 + 2q{v'}_1^1 - 2m\frac{\Omega_2}{\rho}D_1^1 + 2d\frac{\Omega_1}{\rho}D_2^1. \end{aligned} \tag{D.162}$$

With (D.155), (D.161) and (D.162) we have:

$$\begin{aligned}
X_{21} &= i\Omega_1 L_1 + \Omega_2 M_1 \\
&= -\Omega_1 (j_2^1 + 2qv_1^1 + 2m\frac{\Omega_1}{\rho} D_1^1 + 2d\frac{\Omega_2}{\rho} D_2^1) \\
&\quad + \Omega_2 ({j'}_2^1 + 2q{v'}_1^1 - 2m\frac{\Omega_2}{\rho} D_1^1 + 2d\frac{\Omega_1}{\rho} D_2^1) \\
&= -\Omega_1 j_2^1 + \Omega_2 {j'}_2^1 + 2q(-\Omega_1 v_1^1 + \Omega_2 {v'}_1^1) - 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho} D_1^1 \\
&= \rho^2(-\mathcal{S}_{(2)}^1 - 2q\mathcal{A}_{(1)}^1) + 2m\rho D_1^1. \qquad \text{(D.163)}
\end{aligned}$$

Again with (D.137) and after simplification we obtain:

$$\begin{aligned}
L_2 &= i[\partial_0(H_R^2 - H_L^2) + \partial_3(E_R^1 - E_L^1) - \partial_1(E_R^3 - E_L^3) \\
&\quad + 2b_0 E_R^2 + 2a_0 E_L^2 - 2b_3 H_R^1 - 2a_3 H_L^1 + 2b_1 H_R^3 + 2a_1 H_L^3] \\
&= i(j_2^2 + 2qv_1^2 + 2m\frac{\Omega_1}{\rho} D_1^2 + 2d\frac{\Omega_2}{\rho} D_2^2). \qquad \text{(D.164)}
\end{aligned}$$

Similarly with (D.137) and simplifying we get:

$$\begin{aligned}
M_2 &= \partial_0(E_R^2 - E_L^2) + \partial_3(-H_R^1 + H_L^1) + \partial_1(H_R^3 - H_L^3) \\
&\quad + 2(-b_0 H_R^2 - b_3 E_R^1 + b_1 E_R^3) + 2(-a_0 H_L^2 - a_3 E_L^1 + a_1 E_L^3) \\
&= {j'}_2^2 + 2q{v'}_1^2 - 2m\frac{\Omega_2}{\rho} D_1^2. \qquad \text{(D.165)}
\end{aligned}$$

With (D.155), (D.164) and (D.165) we have:

$$\begin{aligned}
X_{22} &= i\Omega_1 L_2 + \Omega_2 M_2 \\
&= -\Omega_1 \Big(j_2^2 + 2qv_1^2 + 2m\frac{\Omega_1}{\rho} D_1^2 + 2d\frac{\Omega_2}{\rho} D_2^2\Big) \\
&\quad + \Omega_2 \Big({j'}_2^2 + 2q{v'}_1^2 - 2m\frac{\Omega_2}{\rho} D_1^2 + 2d\frac{\Omega_1}{\rho} D_2^2\Big) \\
&= -\Omega_1 j_2^2 + \Omega_2 {j'}_2^2 + 2q(-\Omega_1 v_1^2 + \Omega_2 {v'}_1^2) - 2m\frac{\Omega_1^2 + \Omega_2^2}{\rho} D_1^2 \\
&= \rho^2(-\mathcal{S}_{(2)}^2 - 2q\mathcal{A}_{(1)}^2) + 2m\rho D_1^2. \qquad \text{(D.166)}
\end{aligned}$$

Again with (D.137) and after simplification we get:

$$\begin{aligned}
L_3 &= i[\partial_0(H_R^3 - H_L^3) + \partial_2(E_R^2 - E_L^2) - \partial_2(E_R^1 - E_L^1) \\
&\quad + 2b_0 E_R^3 + 2a_0 E_L^3 - 2b_1 H_R^2 - 2a_1 H_L^2 + 2b_2 H_R^1 + 2a_2 H_L^1] \\
&= i\Big(j_2^3 + 2qv_1^3 + 2m\frac{\Omega_1}{\rho} D_1^3 + 2d\frac{\Omega_2}{\rho} D_2^3\Big). \qquad \text{(D.167)}
\end{aligned}$$

Similarly with (D.137) and simplifying we have:

$$\begin{aligned}M_3 &= \partial_0(E^3_R - E^3_L) + \partial_1(-H^2_R + H^2_L) + \partial_2(H^1_R - H^1_L)\\ &\quad + 2(-b_0 H^3_R - b_1 D^2_R + b_2 E^1_R) + 2(-a_0 H^3_L - a_1 E^2_L + a_2 E^1_L)\\ &= {j'}^3_2 + 2q{v'}^3_1 - 2m\frac{\Omega_2}{\rho}D^3_1. \end{aligned}\tag{D.168}$$

With (D.155), (D.167) and (D.168) we obtain:

$$\begin{aligned}X_{23} &= i\Omega_1 L_3 + \Omega_2 M_3\\ &= -\Omega_1\Big(j^3_2 + 2qv^3_1 + 2m\frac{\Omega_1}{\rho}D^3_1 + 2d\frac{\Omega_2}{\rho}D^3_2\Big)\\ &\quad + \Omega_2\Big({j'}^3_2 + 2q{v'}^3_1 - 2m\frac{\Omega_2}{\rho}D^3_1 + 2d\frac{\Omega_1}{\rho}D^3_2\Big)\\ &= -\Omega_1 j^3_2 + \Omega_2 {j'}^3_2 + 2q(-\Omega_1 v^3_1 + \Omega_2 {v'}^3_1) - 2m\frac{\Omega_1^2+\Omega_2^2}{\rho}D^3_1\\ &= \rho^2(-\mathcal{S}^3_{(2)} - 2q\mathcal{A}^3_{(1)}) + 2m\rho D^3_1. \end{aligned}\tag{D.169}$$

And we thus have:

$$\begin{aligned}\Gamma^0_{2\nu} &= \rho^{-2}D^{\mu}_{\nu}X_{2\mu} = \rho^{-2}\Big[D^0_\nu X_{20} + \sum_{k=1}^{3} D^k_\nu X_{2k}\Big]\\ &= \rho^{-2}[D^0_\nu(\rho^2\mathcal{S}^0_{(2)} + 2q\rho^2\mathcal{A}^0_{(1)} + 2m\rho D^0_1)\\ &\quad + \sum_{k=1}^{3} D^k_\nu(-\rho^2\mathcal{S}^k_{(2)} - 2q\rho^2\mathcal{A}^k_{(1)} - 2m\rho D^k_1)]\\ &= D_\nu\cdot\Big(\mathcal{S}_{(2)} + 2q\mathcal{A}_{(1)} + 2\frac{m}{\rho}D_1\Big)\\ &= D_\nu\cdot(\mathcal{S}_{(2)} + 2q\mathcal{A}_{(1)}) - 2m\rho\delta^1_\nu, \end{aligned}\tag{D.170}$$

which is (4.34),

D.4.6 Calculation of $\Gamma^0_{3\nu}$

We begin with:

$$\Gamma^0_{3\nu} = \rho^{-2}D^{\mu}_{\nu}X_{3\mu}, \tag{D.171}$$

$$X_{3\mu} = -D^0_3\partial_\mu D^0_0 + D^1_3\partial_\mu D^1_0 + D^2_3\partial_\mu D^2_0 + D^3_3\partial_\mu D^3_0. \tag{D.172}$$

We have:

$$\begin{aligned}X_{3\mu} &= -(\xi_1\xi_1^* + \xi_2\xi_2^* - \eta_1\eta_1^* - \eta_2\eta_2^*)\partial_\mu(\xi_1\xi_1^* + \xi_2\xi_2^* + \eta_1\eta_1^* + \eta_2\eta_2^*)\\ &\quad + (\xi_1\xi_2^* + \xi_2\xi_1^* + \eta_1\eta_2^* + \eta_2\eta_1^*)\partial_\mu(\xi_1\xi_2^* + \xi_2\xi_1^* - \eta_1\eta_2^* - \eta_2\eta_1^*)\\ &\quad + i(\xi_1\xi_2^* - \xi_2\xi_1^* + \eta_1\eta_2^* - \eta_2\eta_1^*)\partial_\mu i(\xi_1\xi_2^* - \xi_2\xi_1^* - \eta_1\eta_2^* + \eta_2\eta_1^*)\\ &\quad + (\xi_1\xi_1^* - \xi_2\xi_2^* + \eta_1\eta_1^* - \eta_2\eta_2^*)\partial_\mu(\xi_1\xi_1^* - \xi_2\xi_2^* - \eta_1\eta_1^* + \eta_2\eta_2^*), \end{aligned}\tag{D.173}$$

which by (A.84) gives:

$$X_{3\mu} = \begin{pmatrix} \eta_1^*(\Omega_1 - i\Omega_2)\partial_\mu\xi_1 + \eta_2^*(\Omega_1 - i\Omega_2)\partial_\mu\xi_2 \\ -\xi_1^*(\Omega_1 + i\Omega_2)\partial_\mu\eta_1 - \xi_2^*(\Omega_1 + i\Omega_2)\partial_\mu\eta_2 \\ +\eta_1(\Omega_1 + i\Omega_2)\partial_\mu\xi_1^* + \eta_2(\Omega_1 + i\Omega_2)\partial_\mu\xi_2^* \\ -\xi_1(\Omega_1 - i\Omega_2)\partial_\mu\eta_1^* - \xi_2(\Omega_1 - i\Omega_2)\partial_\mu\eta_2^* \end{pmatrix}. \tag{D.174}$$

We let:

$$X_{3\mu} = \Omega_1 N_\mu + i\Omega_2 P_\mu, \tag{D.175}$$

$$\begin{aligned} N_\mu &= \eta_1^*\partial_\mu\xi_1 + \eta_2^*\partial_\mu\xi_2 - \xi_1^*\partial_\mu\eta_1 - \xi_2^*\partial_\mu\eta_2 \\ &\quad + \eta_1\partial_\mu\xi_1^* + \eta_2\partial_\mu\xi_2^* - \xi_1\partial_\mu\eta_1^* - \xi_2\partial_\mu\eta_2^*, \end{aligned} \tag{D.176}$$

$$\begin{aligned} P_\mu &= -\eta_1^*\partial_\mu\xi_1 - \eta_2^*\partial_\mu\xi_2 - \xi_1^*\partial_\mu\eta_1 - \xi_2^*\partial_\mu\eta_2 \\ &\quad + \eta_1\partial_\mu\xi_1^* + \eta_2\partial_\mu\xi_2^* + \xi_1\partial_\mu\eta_1^* + \xi_2\partial_\mu\eta_2^*. \end{aligned} \tag{D.177}$$

With (D.137) and after simplification we get:

$$\begin{aligned} N_0 &= -\partial_1 E_3^1 - \partial_2 E_3^2 - \partial_3 E_3^3 + (b_0 + a_0)\Omega_2 \\ &\quad + (b_1 - a_1)H_3^1 + (b_2 - a_2)H_3^2 + (b_3 - a_3)H_3^3 \\ &= j_3^0 + 2\Omega_2\Big(qA^0 + m\mathrm{v}^0 + \frac{d}{\rho}\mathrm{D}_3^0\Big). \end{aligned} \tag{D.178}$$

Again with (D.137) and after simplification we get:

$$\begin{aligned} P_0 &= \partial_1(iH_3^1) + \partial_2(iH_3^2) + \partial_3(iH_3^3) + i(b_0 + a_0)\Omega_1 \\ &\quad + i(b_1 - a_1)E_3^1 + i(b_2 - a_2)E_3^2 + i(b_3 - a_3)E_3^3 \\ &= i\Big[{j'}_3^0 + 2\Omega_1\Big(qA_0 + m\mathrm{v}_0 + \frac{d}{\rho}D_3^0\Big)\Big]. \end{aligned} \tag{D.179}$$

In light of (D.175), (D.178) and (D.179) taken together we get:

$$\begin{aligned} X_{30} &= \Omega_1 N_0 + \Omega_2 i P_0 \\ &= \Omega_1\Big[j_3^0 + 2\Omega_2\Big(qA^0 + m\mathrm{v}^0 + \frac{d}{\rho}\mathrm{D}_3^0\Big)\Big] - \Omega_2\Big[{j'}_3^0 + 2\Omega_1\Big(qA_0 + m\mathrm{v}_0 + \frac{d}{\rho}D_3^0\Big)\Big] \\ &= \Omega_1 j_3^0 - \Omega_2 {j'}_3^0 \\ &= \rho^2\mathcal{S}_{(3)}^0. \end{aligned} \tag{D.180}$$

As always with (D.137) and after simplification we get:

$$\begin{aligned} N_1 &= -\partial_0 E_3^1 + \partial_2 H_3^3 - \partial_3 H_3^2 \\ &\quad + (b_1 + a_1)\Omega_2 + (b_0 - a_0)H_3^1 + (b_2 - a_2)E_3^3 + (b_3 - a_3)(-E_3^2) \\ &= -j_3^1 - 2\Omega_2\Big(qA^1 + \frac{m}{\rho}\mathrm{D}_0^1 + \frac{d}{\rho}\mathrm{D}_3^1\Big). \end{aligned} \tag{D.181}$$

With (D.137) and after simplification we get:

$$\begin{aligned}P_1 &= i(\partial_0 H^1_3 + \partial_2 E^3_3 - \partial_3 E^2_3 + (b_1 + a_1)E^1_3\\ &\quad + (b_0 - a_0)E^1_3 - (b_2 - a_2)H^3_3 + (b_3 - a_3)H^2_3\\ &= i\Big[- {j'}^1_3 - 2\Omega_1\Big(qA^1 + \frac{m}{\rho}D^1_0 + \frac{d}{\rho}D^1_3\Big)\Big].\end{aligned} \tag{D.182}$$

With (D.175), (D.181) and (D.182) we get:

$$\begin{aligned}X_{31} &= \Omega_1 N_1 + \Omega_2 i P_1\\ &= \Omega_1\Big[- j^1_3 - 2\Omega_2\Big(qA^1 + \frac{m}{\rho}D^1_0 + \frac{d}{\rho}D^1_3\Big)\Big]\\ &\quad - \Omega_2\Big[- {j'}^1_3 - 2\Omega_1\Big(qA^1 + \frac{m}{\rho}D^1_0 + \frac{d}{\rho}D^1_3\Big)\Big]\\ &= -\Omega_1 j^1_3 + \Omega_2 {j'}^1_3 = -\rho^2 \mathcal{S}^1_{(3)}.\end{aligned} \tag{D.183}$$

Again with (D.137) we get:

$$\begin{aligned}N_2 &= -\partial_0 E^2_3 - \partial_1 H^3_3 + \partial_3 H^1_3 + (b_2 + a_2)\Omega_2\\ &\quad + (b_0 - a_0)H^2_3 + (b_3 - a_3)E^1_3 - (b_1 - a_1)E^2_3\\ &= -j^2_3 - 2\Omega_2\Big(qA^2 + \frac{m}{\rho}D^2_0 + \frac{d}{\rho}D^2_3\Big).\end{aligned} \tag{D.184}$$

As usual with (D.137) we obtain:

$$\begin{aligned}P_2 &= i(\partial_0 H^2_3 + \partial_3 E^1_3 - \partial_1 E^3_3 + (b_2 + a_2)E^2_3\\ &\quad + (b_0 - a_0)E^2_3 - (b_3 - a_3)H^1_3 + (b_1 - a_1)H^3_3\\ &= i\Big[- {j'}^2_3 - 2\Omega_1\Big(qA^2 + \frac{m}{\rho}D^2_0 + \frac{d}{\rho}D^2_3\Big)\Big].\end{aligned} \tag{D.185}$$

With (D.175), (D.184) and (D.185) we have:

$$\begin{aligned}X_{32} &= \Omega_1 N_2 + \Omega_2 i P_2\\ &= \Omega_1\Big[- j^2_3 - 2\Omega_2\Big(qA^2 + \frac{m}{\rho}D^2_0 + \frac{d}{\rho}D^2_3\Big)\Big]\\ &\quad - \Omega_2\Big[- {j'}^2_3 - 2\Omega_1\Big(qA^2 + \frac{m}{\rho}D^2_0 + \frac{d}{\rho}D^2_3\Big)\Big]\\ &= -\Omega_1 j^2_3 + \Omega_2 {j'}^2_3 = -\rho^2 \mathcal{S}^2_{(3)}.\end{aligned} \tag{D.186}$$

Always with (D.137) we get:

$$\begin{aligned}N_3 &= -\partial_0 E^3_3 - \partial_2 H^1_3 + \partial_1 H^2_3 + (b_3 + a_3)\Omega_2\\ &\quad + (b_0 - a_0)H^3_3 + (b_1 - a_1)E^2_3 - (b_2 - a_2)E^3_3\\ &= -j^3_3 - 2\Omega_2(qA^3 + \frac{m}{\rho}D^3_0 + \frac{d}{\rho}D^3_3).\end{aligned} \tag{D.187}$$

With (D.137) and after simplification we have:

$$\begin{aligned} P_3 &= i(\partial_0 H^3_3 + \partial_1 E^2_3 - \partial_2 E^1_3 + (b_3 + a_3)E^3_3 \\ &\quad + (b_0 - a_0)E^3_3 - (b_1 - a_1)H^2_3 + (b_2 - a_2)H^1_3 \\ &= i\Big[- {j'}^3_3 - 2\Omega_1\Big(qA^3 + \frac{m}{\rho}D^3_0 + \frac{d}{\rho}D^3_3\Big)\Big]. \end{aligned} \tag{D.188}$$

Hence with (D.175), (D.187) and (D.188) we get:

$$\begin{aligned} X_{33} &= \Omega_1 N_3 + \Omega_2 i P_3 \\ &= \Omega_1\Big[- j^3_3 - 2\Omega_2\Big(qA^3 + \frac{m}{\rho}D^3_0 + \frac{d}{\rho}D^3_3\Big)\Big] \\ &\quad - \Omega_2\Big[- {j'}^3_3 - 2\Omega_1\Big(qA^3 + \frac{m}{\rho}D^3_0 + \frac{d}{\rho}D^3_3\Big)\Big] \\ &= -\Omega_1 j^3_3 + \Omega_2 {j'}^3_3 = -\rho^2 \mathcal{S}^3_{(3)}. \end{aligned} \tag{D.189}$$

And we thus have:

$$\begin{aligned} \Gamma^0_{3\nu} &= \rho^{-2} D^\mu_\nu X_{3\mu} = \rho^{-2}\Big[D^0_\nu X_{30} + \sum_{k=1}^{3} D^k_\nu X_{3k}\Big] \\ &= \rho^{-2}\Big[D^0_\nu\Big(\rho^2\mathcal{S}^0_{(3)} + \sum_{k=1}^{3} D^k_\nu(-\rho^2\mathcal{S}^k_{(3)})\Big)\Big] \\ &= D_\nu \cdot \mathcal{S}_{(3)}, \end{aligned} \tag{D.190}$$

which is (4.35).

D.4.7 Calculation of $\Gamma^k_{l\nu}$

We must calculate these symbols for $k = 1,\ 2,\ 3$; $l = 1,\ 2,\ 3$ and $l \neq k$. We start from:

$$\begin{aligned} \Gamma^k_{l\nu} &= \rho^{-2}(\boldsymbol{\partial}_\nu D^\mu_l)\overline{D}^k_\mu \\ &= \rho^{-2}\Big[(\boldsymbol{\partial}_\nu D^0_l)\overline{D}^k_0 + \sum_{n=1}^{3}(\boldsymbol{\partial}_\nu D^n_l)\overline{D}^k_n\Big] \\ &= \rho^{-2}\Big[(\boldsymbol{\partial}_\nu D^0_l)(-D^0_k) + \sum_{n=1}^{3}(\boldsymbol{\partial}_\nu D^n_l)D^n_k\Big], \end{aligned} \tag{D.191}$$

and similarly:

$$
\begin{aligned}
\Gamma^l_{k\nu} &= \rho^{-2}(\boldsymbol{\partial}_\nu D^\mu_k)\overline{D}^l_\mu \\
&= \rho^{-2}\Big[(\boldsymbol{\partial}_\nu D^0_k)\overline{D}^l_0 + \sum_{n=1}^{3}(\boldsymbol{\partial}_\nu D^n_k)\overline{D}^l_n\Big] \\
&= \rho^{-2}\Big[(\boldsymbol{\partial}_\nu D^0_k)(-D^0_l) + \sum_{n=1}^{3}(\boldsymbol{\partial}_\nu D^n_k)D^n_l\Big]. \qquad \text{(D.192)}
\end{aligned}
$$

We thus get:

$$
\begin{aligned}
\Gamma^k_{l\nu} + \Gamma^l_{k\nu} &= \rho^{-2}\Big[-\boldsymbol{\partial}_\nu(D^0_k D^0_l) + \sum_{n=1}^{3}\boldsymbol{\partial}_\nu(D^n_k D^n_l)\Big], \\
&= -\rho^{-2}\boldsymbol{\partial}_\nu(D_k \cdot D_l) = 0 \qquad \text{(D.193)} \\
\Gamma^l_{k\nu} &= -\Gamma^k_{l\nu}. \qquad \text{(D.194)}
\end{aligned}
$$

The calculation of $\Gamma^1_{2\nu}$, $\Gamma^2_{3\nu}$ and $\Gamma^3_{1\nu}$ is thus sufficient. Moreover we have:

$$
\begin{aligned}
\rho^2\Gamma^k_{l\nu} &= \begin{pmatrix}
-D^0_k(D^0_\nu\partial_0 + D^1_\nu\partial_1 + D^2_\nu\partial_2 + D^3_\nu\partial_3)(D^0_l) \\
+D^1_k(D^0_\nu\partial_0 + D^1_\nu\partial_1 + D^2_\nu\partial_2 + D^3_\nu\partial_3)(D^1_l) \\
+D^2_k(D^0_\nu\partial_0 + D^1_\nu\partial_1 + D^2_\nu\partial_2 + D^3_\nu\partial_3)(D^2_l) \\
+D^3_k(D^0_\nu\partial_0 + D^1_\nu\partial_1 + D^2_\nu\partial_2 + D^3_\nu\partial_3)(D^3_l)
\end{pmatrix} \\
&= \begin{pmatrix}
D^0_\nu(-D^0_k\partial_0 D^0_l + D^1_k\partial_0 D^1_l + D^2_k\partial_0 D^2_l + D^3_k\partial_0 D^3_l) \\
+D^1_\nu(-D^0_k\partial_1 D^0_l + D^1_k\partial_1 D^1_l + D^2_k\partial_1 D^2_l + D^3_k\partial_1 D^3_l) \\
+D^2_\nu(-D^0_k\partial_2 D^0_l + D^1_k\partial_2 D^1_l + D^2_k\partial_2 D^2_l + D^3_k\partial_2 D^3_l) \\
+D^3_\nu(-D^0_k\partial_3 D^0_l + D^1_k\partial_3 D^1_l + D^2_k\partial_3 D^2_l + D^3_k\partial_3 D^3_l)
\end{pmatrix} \\
&= D^\mu_\nu(-D^0_k\partial_\mu D^0_l + D^1_k\partial_\mu D^1_l - D^2_k\partial_\mu D^2_l + D^3_k\partial_\mu D^3_l). \qquad \text{(D.195)}
\end{aligned}
$$

Calculation of $\Gamma^1_{2\nu}$

Given that:

$$
\begin{aligned}
\rho^2\Gamma^1_{2\nu} &= D^\mu_\nu(iW_\mu), \\
iW_\mu &= -D^0_1\partial_\mu D^0_2 + D^1_1\partial_\mu D^1_2 - D^2_1\partial_\mu D^2_2 + D^3_1\partial_\mu D^3_2 \qquad \text{(D.196)} \\
&= -(-\xi_1^*\eta_2^* - \xi_1\eta_2 + \xi_2^*\eta_1^* + \xi_2\eta_1)\partial_\mu i(-\xi_1^*\eta_2^* + \xi_1\eta_2 + \xi_2^*\eta_1^* - \xi_2\eta_1) \\
&+ (\xi_1^*\eta_1^* - \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1)\partial_\mu i(\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* - \xi_1\eta_1) \\
&+ i(-\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1)\partial_\mu(\xi_1^*\eta_1^* + \xi_2\eta_2 + \xi_2^*\eta_2^* + \xi_1\eta_1) \\
&+ (-\xi_1^*\eta_2^* - \xi_1\eta_2 - \xi_2^*\eta_1^* - \xi_2\eta_1)\partial_\mu i(-\xi_1^*\eta_2^* + \xi_1\eta_2 - \xi_2^*\eta_1^* + \xi_2\eta_1).
\end{aligned}
$$

We thus have:

$$\begin{aligned}\frac{1}{2}W_\mu &= (\partial_\mu\xi_1)(-\eta_1^*)(\eta_2\xi_2^* + \eta_1\xi_1^*) + (\partial_\mu\xi_2)(-\eta_2^*)(\eta_1\xi_1^* + \eta_2\xi_2^*)\\ &+ (\partial_\mu\eta_1)(-\xi_1^*)(\xi_2\eta_2^* + \xi_1\eta_1^*) + (\partial_\mu\eta_2)(-\xi_2^*)(\xi_1\eta_1^* + \xi_2\eta_2^*)\\ &+ (\partial_\mu\xi_1^*)\eta_1(\xi_2\eta_2^* + \xi_1\eta_1^*) + (\partial_\mu\xi_2^*)\eta_2(\xi_1\eta_1^* + \xi_2\eta_2^*)\\ &+ (\partial_\mu\eta_1^*)\xi_1(\eta_2\xi_2^* + \eta_1\xi_1^*) + (\partial_\mu\eta_2^*)\xi_2(\eta_1\xi_1^* + \eta_2\xi_2^*), \end{aligned}\tag{D.197}$$

which gives with equations (D.176) and(D.177) calculating N_μ and P_μ:

$$iW_\mu = \Omega_1 iP_\mu - \Omega_2 N_\mu. \tag{D.198}$$

We may thus use the results of D.4.6 and we directly obtain:

$$\begin{aligned}\rho^2\Gamma^1_{2\nu} &= D^0_\nu\Big[-\rho^2\mathcal{S}'^0_{(3)} - 2\rho^2\Big(qA^0 + \frac{m}{\rho}\mathrm{D}^0_0 + \frac{d}{\rho}\mathrm{D}^0_3\Big)\Big]\\ &\quad + D^k_\nu\Big[\rho^2\mathcal{S}'^k_{(3)} + 2\rho^2\Big(qA^k + \frac{m}{\rho}\mathrm{D}^k_0 + \frac{d}{\rho}\mathrm{D}^k_3\Big)\Big]\\ \Gamma^1_{2\nu} &= -D_\nu\cdot(\mathcal{S}'_{(3)} + 2qA) - 2m\rho\delta^0_\nu + 2d\rho\delta^3_\nu. \end{aligned}\tag{D.199}$$

which gives (4.38).

Calculation of $\Gamma^2_{3\nu}$

Similarly we have:

$$\begin{aligned}\rho^2\Gamma^2_{3\nu} &= D^\mu_\nu R_\mu,\\ R_\mu &= -D^0_2\partial_\mu D^0_3 + D^1_2\partial_\mu D^1_3 + D^2_2\partial_\mu D^2_3 + D^3_2\partial_\mu D^3_3 \qquad \text{(D.200)}\\ &= -i(-\xi_1^*\eta_2^* + \xi_1\eta_2 + \xi_2^*\eta_1^* - \xi_2\eta_1)\partial_\mu(\xi_1\xi_1^* + \xi_2\xi_2^* - \eta_1\eta_1^* - \eta_2\eta_2^*)\\ &+ i(\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* - \xi_1\eta_1)\partial_\mu(\xi_1\xi_2^* + \xi_2\xi_1^* + \eta_1\eta_2^* + \eta_2\eta_1^*)\\ &+ (\xi_1^*\eta_1^* + \xi_2\eta_2 + \xi_2^*\eta_2^* + \xi_1\eta_1)\partial_\mu i(\xi_1\xi_2^* - \xi_2\xi_1^* + \eta_1\eta_2^* - \eta_2\eta_1^*)\\ &+ i(-\xi_1^*\eta_2^* + \xi_1\eta_2 - \xi_2^*\eta_1^* + \xi_2\eta_1)\partial_\mu(\xi_1\xi_1^* - \xi_2\xi_2^* + \eta_1\eta_1^* - \eta_2\eta_2^*). \end{aligned}\tag{D.201}$$

Using always (D.137) we have:

$$R_\mu = i\begin{pmatrix} \xi_2(\partial_\mu\xi_1)(\Omega_1 - i\Omega_2) - \xi_1(\partial_\mu\xi_2)(\Omega_1 - i\Omega_2)\\ +\eta_2(\partial_\mu\eta_1)(\Omega_1 + i\Omega_2) - \eta_1(\partial_\mu\eta_2)(\Omega_1 + i\Omega_2)\\ -\xi_2^*(\partial_\mu\xi_1^*)(\Omega_1 + i\Omega_2) + \xi_1^*(\partial_\mu\xi_2^*)(\Omega_1 + i\Omega_2)\\ -\eta_2^*(\partial_\mu\eta_1^*)(\Omega_1 - i\Omega_2) + \eta_1^*(\partial_\mu\eta_2^*)(\Omega_1 - i\Omega_2)\end{pmatrix}. \tag{D.202}$$

With equations (D.134) and (D.135) calculating Y_μ and Z_μ we deduce:

$$R_\mu = -\Omega_2 Y_\mu + \Omega_1 iZ_\mu. \tag{D.203}$$

We may then use the results in D.4.4 and we directly obtain:

$$\begin{aligned}\rho^2\Gamma^2_{3\nu} &= \mathrm{D}^{\mu}_{\nu}[-(\Omega_2 j_{1\mu} + \Omega_1 j_{1\mu} - 2q(\Omega_2 v^{\mu}_2 + \Omega_1 v'^{\mu}_2)\\ &\quad - 2m\frac{\Omega_1\Omega_2 - \Omega_2\Omega_1}{\rho}\mathrm{D}^{\mu}_2 + 2d\frac{\Omega_1^2 + \Omega_2^2}{\rho}\mathrm{D}^{\mu}_1]\\ \Gamma^2_{3\nu} &= -\mathrm{D}_{\nu}\cdot(\mathcal{S}'_{(1)} + 2q\mathcal{A}'_{(2)}) - 2d\rho\delta^1_{\nu}. \end{aligned} \tag{D.204}$$

This is (4.36).

Calculation of $\Gamma^3_{1\nu}$

We finally have:

$$\begin{aligned}\rho^2\Gamma^3_{1\nu} &= \mathrm{D}^{\mu}_{\nu}(Q_{\mu}),\\ Q_{\mu} &= -\mathrm{D}^0_3\partial_{\mu}\mathrm{D}^0_1 + \mathrm{D}^1_3\partial_{\mu}\mathrm{D}^1_1 + \mathrm{D}^2_3\partial_{\mu}\mathrm{D}^2_1 + \mathrm{D}^3_3\partial_{\mu}\mathrm{D}^3_1 && \text{(D.205)}\\ &= -(\xi_1\xi_1^* + \xi_2\xi_2^* - \eta_1\eta_1^* - \eta_2\eta_2^*)\partial_{\mu}(-\xi_1^*\eta_2^* - \xi_1\eta_2 + \xi_2^*\eta_1^* + \xi_2\eta_1)\\ &\quad + (\xi_1\xi_2^* + \xi_2\xi_1^* + \eta_1\eta_2^* + \eta_2\eta_1^*)\partial_{\mu}(\xi_1^*\eta_1^* - \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1)\\ &\quad + i^2(\xi_1\xi_2^* - \xi_2\xi_1^* + \eta_1\eta_2^* - \eta_2\eta_1^*)\partial_{\mu}(-\xi_1^*\eta_1^* + \xi_2\eta_2 - \xi_2^*\eta_2^* + \xi_1\eta_1)\\ &\quad + (\xi_1\xi_1^* - \xi_2\xi_2^* + \eta_1\eta_1^* - \eta_2\eta_2^*)\partial_{\mu}(-\xi_1^*\eta_2^* - \xi_1\eta_2 - \xi_2^*\eta_1^* - \xi_2\eta_1). && \text{(D.206)}\end{aligned}$$

We get with (D.156) and (D.157) calculating L_{μ} and M_{μ}:

$$Q_{\mu} = \Omega_1 M_{\mu} - i\Omega_2 L_{\mu}. \tag{D.207}$$

We may thus use the results of D.4.5 and we directly get:

$$\begin{aligned}\rho^2\Gamma^3_{1\nu} &= \mathrm{D}^0_{\nu}[\Omega_2(-j^0_2 - 2qv^0_1 - 2m\frac{\Omega_1}{\rho}\mathrm{D}^0_1 - 2d\frac{\Omega_2}{\rho}\mathrm{D}^0_2)\\ &\quad + \Omega_1(-j'^0_2 - 2qv'^0_1 + 2m\frac{\Omega_2}{\rho}\mathrm{D}^0_1 - 2d\frac{\Omega_1}{\rho}\mathrm{D}^0_2)]\\ &\quad + \sum_{k=1}^{3}\mathrm{D}^k_{\nu}[\Omega_2(j^k_2 + 2qv^k_1 + 2m\frac{\Omega_1}{\rho}\mathrm{D}^k_2 + 2d\frac{\Omega_2}{\rho}\mathrm{D}^k_2)\\ &\quad + \Omega_1(j'^k_2 + 2qv'^k_1 - 2m\frac{\Omega_2}{\rho}\mathrm{D}^k_1 + 2d\frac{\Omega_1}{\rho}\mathrm{D}^k_2))].\end{aligned} \tag{D.208}$$

This gives:

$$\Gamma^3_{1\nu} = \mathrm{D}_{\nu}\cdot(-\mathcal{S}'_{(2)} - 2q\mathcal{A}'_{(1)}) + 2d\rho\delta^2_{\nu}, \tag{D.209}$$

which is (4.37), end of this long and tedious calculation.

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